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Limit Order Market under Asymmetric Information

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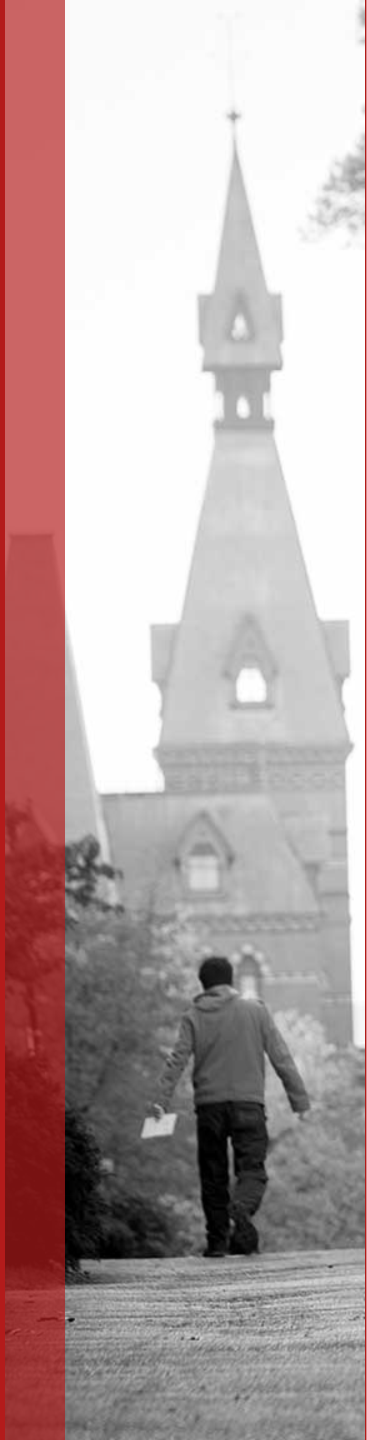


Motivation

- The adverse selection paradigm has transformed market microstructure.
- Useful frameworks used a dealer market structure.
  - Glosten-Milgrom-Easley-O'Hara and Kyle (1985).
- Markets have evolved over the past two decades and most exchanges operate a limit order market.
- It is very difficult to model a dynamic limit order market under asymmetric information.
  - Models resort to numerical solutions.
- Our goal is to develop a simple dynamic limit order book under asymmetric information that can be solved analytically.
  - Glosten-Milgrom-Easley-O'Hara meets Rosu (2009).
  - Gives us new insights and a useful framework that can be easily extended.

Literature

- Static limit order book models with asymmetric information:
  - Glosten (1994), Rock (1994), Seppi (1997).
- Dynamic limit order book models without asymmetric information:
  - Parlour (1998), Foucault (1999), Foucault, Kadan, and Kandel (2005), Goettler, Parlour, and Rajan (2005), Rosu (2009), Buti and Rindi (2013), Buti, Rindi, and Werner (2017).
- Hybrid models with asymmetric information:
  - Kaniel and Liu (2006), Brolley and Malinova (2020).
- Dynamic limit order book model with asymmetric information that require numerical solutions:
  - Goettler, Parlour, and Rajan (20009), Ricco, Rindi, and Seppi (2018), Rosu (2020).

The Model

- Market is organized as a continuous-time limit order book.
  - Traders can submit limit orders that enter the book or execute limit orders that reside in the book by trading against them with “market” orders.
  - All orders are for one unit.
  - Usual priority rules: a limit sell order at a lower price is executed before a limit sell order at a higher price, etc.
  - Continuous price grid.
  - Pre- and post-trade transparency.
  - Orders are anonymous.
- Single risky asset.
  - Liquidation value can be either $v+\sigma$ or $v-\sigma$.
  - Expected value of the asset at *t=0* is $v_0 = q_0(v+\sigma) + (1-q_0)(v-\sigma)$.

Traders

- Uninformed traders: exogenous trading need (buy or sell) for hedging or liquidity needs.
- Impatient uninformed traders:
  - Buyers (bi) and sellers (si).
  - Use market orders.
  - Buyers have outside options at prices just above $v + \sigma$ and sellers at prices just below $v - \sigma$.
- Patient uninformed traders:
  - Buyers (bp): $EU_t^{bp} = E_t[\tilde{v} - \pi_T - r(T - t)]$
  - Sellers (sp) $EU_t^{sp} = E_t[\pi_T - \tilde{v} - r(T - t)]$
  - Buyers have outside options at prices just above v_t and sellers at prices just below v_t.
    - Ascribe economic content to their “patience”: they would not wish to trade or wait in the book if their orders would result in an expected loss.
    - Implies that patient traders use limit orders.
    - They would need to choose the limit order price.

Traders– cont.

- Patient informed traders:
  - Do not have an exogenous need to trade (or a private value).
  - Know the realization of the random variable v, and can trade to exploit their information.
  - Expected utility: $EU_t^I = \phi v - E_t[\phi \pi_t - r(T-t)]$ where $\phi = +1$ for a buy order and $\phi = -1$ for a sell order.
  - Choose whether to buy/sell, submit limit/market orders, and limit order price if submit a limit order.
- Utilities of all market participants: $u = \left(u_{bi}, u_{si}, u_{bp}, u_{sp}, u_I\right)$
- This is a model of trading a risky asset. If everyone knows v, traders only agree to trade at v.
- Traders arrive according to independent Poisson processes with constant intensities:

$$\Lambda = \left(\lambda_{bi}, \lambda_{si}, \lambda_{bp}, \lambda_{sp}, \lambda_I\right)$$

Equilibrium Concept

- If you take out the informed traders, our specification of the limit order environment follows closely that of Rosu (2009).
  - The notion of equilibrium we use mirrors the Markov perfect equilibrium framework in Rosu's setup.
- Our setup can be viewed as being comprised of two games:
  - the limit order market game at time t: G_t
  - the broader infinite horizon game: $G = (G_t)_{t \in [0,+\infty)}$
- Notation:
  - x_t^n is the limit sell order with the nth lowest execution priority. All limit sell orders in the book: $x_t = (x_t^1, x_t^2, \ldots)$
  - y_t^n is the limit buy order with the nth lowest execution priority. All limit buy orders in the book: $y_t = (y_t^1, y_t^2, \ldots)$
  - $(p_i)_{i \in x_t}$ are the prices of the limit sell orders; $(p_k)_{k \in y_t}$ are the prices of the limit buy orders.

The Limit Order Market Game G_t

- The limit order market game at time t, G_t, can be represented as:

$$G_t = \left(\Lambda, v_t, x_t, y_t, (p_i)_{i \in x_t}, (p_k)_{k \in y_t}, u\right)$$

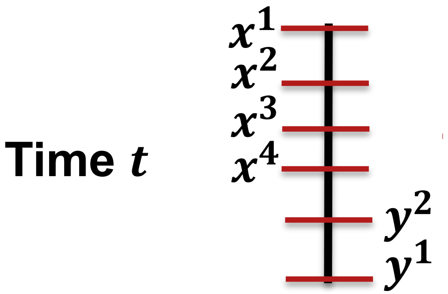
- Players in G_t: patient uninformed traders and informed traders who are present in the market at time t (represented by x_t and y_t).
  - Traders who already have executed their order or traders who have yet to arrive in the market are not players in this game.
  - Since the trading decision of the impatient uninformed traders are exogenous, they are part of the description of the game specified by the arrival rates Λ.
- Actions of players:
  - Patient uninformed traders and informed traders choose to submit limit orders with prices $(p_i)_{i \in x_t}, (p_k)_{k \in y_t}$.
  - Patient uninformed traders can choose instead to take their outside options.
  - Informed traders can choose instead to submit market orders.

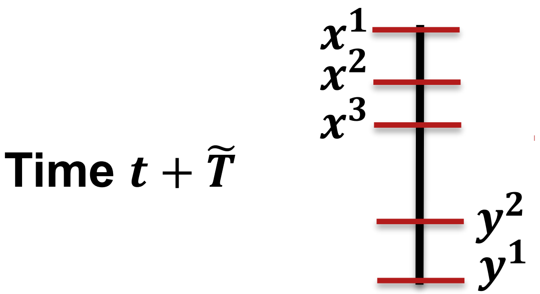
Definition: Equilibrium in Games G_t and G

- An equilibrium in the limit order market game at time t, G_t, is:
  - a set of limit order prices $(p_i)_{i \in x_t}$ and $(p_k)_{k \in y_t}$,
  - informed traders choice of order direction and order type and uninformed traders choice between submitting orders and their outside options such that, given common knowledge of rationality among the players,
    - No patient uninformed trader has an incentive to reprice his limit order.
    - No patient informed trader has an incentive to reprice her limit order.
    - No patient uninformed trader has an incentive to change their decision whether to take the outside option.
    - No patient informed trader has an incentive to change her order choice or her order direction.
- An equilibrium in the infinite horizon game G is a sequence $(G_t)_{t \in [0, +\infty)}$ such that for any $t'' > t'$, $v_{t''} \in G_{t''}$ is obtained from $v_{t'} \in G_{t'}$ using a Bayesian update process given the orders submitted in the interval $[t', t'')$, and traders can cancel and resubmit their orders in accordance with the Bayesian update process.

Wait Time

- A wait time denotes how long an order at a particular position in the execution hierarchy is expected to wait before execution, given a configuration of the book.
- Example: given a configuration $(x_t^1, x_t^2, x_t^3, x_t^4; y_t^1, y_t^2)$, the wait time for the limit order with the highest execution priority on the sell side is a real value function that measures the expected value of the (random) time $\tilde{T}$ it takes for the book to move to the configuration $(x_{t+\tilde{T}}^1, x_{t+\tilde{T}}^2, x_{t+\tilde{T}}^3; y_{t+\tilde{T}}^1, y_{t+\tilde{T}}^2)$. We write this wait time as $w_t(x^4|x^4; y^2)$.





- General notation for a wait time of a limit order at a particular position in the book: $w_t(x^m|x^n; y^k), m \leq n$, is the expected time it takes for the limit sell order with the *m*th lowest execution priority to execute when there are *n* limit sell orders and *k* limit buy orders in the book.

Limit Order Market without Asymmetric Information

- When there are no informed traders in the market, the wait times do not depend on *t* and the configuration of the other side of the book doesn't matter, so just for the case of limit orders without asymmetric information we simplify the notation to be $w(x^m|x^n)$.
- Consider $w(x^1|x^1)$, which is the wait time until the execution of a limit sell order when it is the only order in the sell side of the book.
- Two things will impact its wait time: arrival of market buy orders that can execute it and arrival of limit sell orders that can undercut it, in which case it cannot execute until the order that undercuts it executes first.
- The next arrival of a trader that will affect the sell side of the book (a patient seller or an impatient buyer) happens after an expected time interval of $\frac{1}{\lambda_{bi}+\lambda_{sp}}$.
- The probability that this is an impatient buyer arrival is $\frac{\lambda_{bi}}{\lambda_{bi}+\lambda_{sp}}$.

Recursive Formulation

- $w(x^1|x^1) = \left[\frac{1}{\lambda_{bi}+\lambda_{sp}}\right]\left[\frac{\lambda_{bi}}{\lambda_{bi}+\lambda_{sp}}\right] +$

$$\left[\frac{1}{\lambda_{bi}+\lambda_{sp}} + w(x^2|x^2) + \frac{1}{\lambda_{bi}+\lambda_{sp}}\right]\left[\frac{\lambda_{sp}}{\lambda_{bi}+\lambda_{sp}} \cdot \frac{\lambda_{bi}}{\lambda_{bi}+\lambda_{sp}}\right] +$$

$$\left[\frac{1}{\lambda_{bi}+\lambda_{sp}} + w(x^2|x^2) + \frac{1}{\lambda_{bi}+\lambda_{sp}} + w(x^2|x^2) + \frac{1}{\lambda_{bi}+\lambda_{sp}}\right]\left[\frac{\lambda_{sp}}{\lambda_{bi}+\lambda_{sp}} \cdot \frac{\lambda_{sp}}{\lambda_{bi}+\lambda_{sp}} \cdot \frac{\lambda_{bi}}{\lambda_{bi}+\lambda_{sp}}\right] +$$

$\ldots$

- This infinite series simplifies to:
- $w(x^1|x^1) = \left[w(x^2|x^2) + \frac{1}{\lambda_{bi}+\lambda_{sp}}\right] \cdot \frac{\lambda_{sp}}{\lambda_{bi}} + \frac{1}{\lambda_{bi}+\lambda_{sp}}$

Closing the Recursive Formulation

- To close the recursive formulation, we use the assumption that patient uninformed traders have outside options.
- If a patient trader arrives in the book and, after considering the price at which he can submit the order and the expected wait time to execute, prefers to take the outside option, we call this situation a full book.
- Definition: the sell (buy) side of the limit order book is full if an arriving patient seller (buyer) prefers his outside option over waiting in the book.
- Notation: $w(x^F|x^F)$ is the wait time of the limit sell order with the highest execution priority when the sell side of the book is full.
- Then, $w(x^{F+1}|x^{F+1}) = 0$, and plugging it into the recursive formula, we get $w(x^F|x^F) = \frac{1}{\lambda_{bi}}$.

Price Gaps

- Sellers (sp) $EU_t^{sp} = E_t[\pi_t - \tilde{v} - (T - t)]$ (with $r = 1$).
- Price gap: Let $g_t(x^n, x^{n+1}|x^{n+m})$ denote the difference in price (or price gap) between the *n*th lowest execution priority sell order and the *(n+1)*th lowest priority sell order when the book has *n+m* orders, $m \geq 1$, at time *t*.
- Proposition 1: $g_t(x^n, x^{n+1}|x^{n+m}) = w(x^n|x^{n+m}) - w(x^{n+1}|x^{n+m}), m \geq 1.$
- The reduction in expected utility from repositioning a limit order at a lower price to gain priority must equal in equilibrium the improvement in wait time.
  - E.g., : $g_t(x^1, x^2|x^2) = w(x^1|x^2) - w(x^2|x^2)$.

Calculating Limit Order Prices

- Proposition 2: The lowest limit sell price when the sell side of the book is full, $p(x^F|x^F)$, satisfies the condition $p(x^F|x^F) \geq v_0 + \frac{1}{\lambda_{bi}}$.
- Proposition 3: In an empty book, a limit order seller prefers a higher to a lower price.
- The boundary conditions in the above two propositions together with the price gaps (Proposition 1) can be used to find the prices of all limit orders in the book.
- For the limit order book without asymmetric information, these limit order prices provide an alternative characterization of the Markov perfect equilibrium described in Rosu (2009).

Limit Order Market under Asymmetric Information

- Like in traditional sequential trade models, the buy and sell orders of the uninformed traders are random selections form a probability distribution that is common knowledge. The informed traders direct their orders according to the value of the asset. This changes the otherwise random arrival rates of buy and sell orders, conveying information to the market.
- But…
  - We have a larger state space (not just the bid and ask prices but all orders in the book).
  - The state space dynamically evolves from the decisions of the traders who previously arrived in the market.
  - The informed trader has more choices (direction, order types, limit order price).
- To construct the equilibrium,
  - We characterize the choices of the informed trader.
  - We describe the uninformed traders' Bayesian update of beliefs about the asset value.

Informed Trader's Strategies

- Proposition 5: When the asset value is low (high) and the sell (buy) side of the limit order book is not full, an informed trader mimics the decisions of a patient seller (buyer).
  - The expected utility of the informed trader from using a limit order when the book is not full is greater than the expected utility from using a market sell order.
  - The informed trader will use the same limit order prices as the patient uninformed traders in order to hide among them.
- Proposition 6: When the asset value Is low (high) and the sell (buy) side of the limit order book is full, an informed trader mimics an impatient seller (buyer).
  - Submitting a limit order will reveal the identity of the informed traders causing immediate adjustment of prices and preventing the informed trader from making money.
- Similar in spirit to Kyle (1985), the strategic informed traders hide by mimicking the strategies (order choices and limit order prices) of the uninformed traders.

Arrival Rates of Orders

- $q_t \equiv P_t[\tilde{v} = v + \sigma]$ is the common knowledge probability at time *t* that the value of the asset is high.
- Let $\lambda_{t,MB}, \lambda_{t,MS}, \lambda_{t,LB}, \lambda_{t,LS}$ denote the common knowledge expected arrival of market buy orders, market sell orders, limit buy orders, and limit sell orders, respectively, at time *t*.
- Corollary 1: At time t, an uninformed trader expects
  - the arrival rate of market buy orders to be $\lambda_{t,MB} = \lambda_{bi} + q_t \lambda_I$ if the buy side of the book is full, and $\lambda_{t,MB} = \lambda_{bi}$ if the buy side of the limit order book is not full.
  - the arrival rate of limit sell orders to be $\lambda_{t,LS} = \lambda_{sp}$ if the sell side of the limit order book is full, and $\lambda_{t,LS} = \lambda_{sp} + (1 - q_t)\lambda_I$ if the sell side of the book is not full. …
- These expected arrival rates can be used to construct the wait times in an analogous fashion to what we did in the limit order book without informed traders.
  - Wait times need the subscript *t* because q_t evolves over time.
  - The entire book's configuration matters for the wait time analysis on either side of the book.

Bayesian Learning from the Order Flow

- The uninformed traders know whether the informed traders use limit orders or market orders. What they don't know is whether the informed traders buy or sell.
- The uninformed traders use the time between order arrivals to update their beliefs about the asset's value.
- Example when the buy side of the book is full:
- $P_t[\tau_{MB}|\tilde{v} = v + \sigma] = (\lambda_{bi} + \lambda_I)e^{-(\lambda_{bi}+\lambda_I)\tau_{MB}}, \ P_t[\tau_{MB}|\tilde{v} = v - \sigma] = \lambda_{bi}e^{-\lambda_{bi}\tau_{MB}}$
- $P_t[\tilde{v} = v + \sigma|\tau_{MB}] = \frac{q_t(\lambda_{bi}+\lambda_I)e^{-(\lambda_{bi}+\lambda_I)\tau_{MB}}}{q_t(\lambda_{bi}+\lambda_I)e^{-(\lambda_{bi}+\lambda_I)\tau_{MB}}+(1-q_t)\lambda_{bi}e^{-\lambda_{bi}\tau_{MB}}}$

$$v_{t+\tau_{MB}}|\tau_{MB} = \frac{(v+\sigma)q_t(\lambda_{bi}+\lambda_I)e^{-(\lambda_{bi}+\lambda_I)\tau_{MB}} + (v-\sigma)(1-q_t)\lambda_{bi}e^{-\lambda_{bi}\tau_{MB}}}{q_t(\lambda_{bi}+\lambda_I)e^{-(\lambda_{bi}+\lambda_I)\tau_{MB}} + (1-q_t)\lambda_{bi}e^{-\lambda_{bi}\tau_{MB}}}$$

Insight: When do Informed Traders Supply or Demand Liquidity?

- The informativeness of arriving market and limit orders depends on the size of the prevailing spread.
- When the spread is small, informed traders demand liquidity by submitting market orders; when the spread is larger, informed traders supply liquidity using limit orders.
  - When the spread is wide, patient uninformed traders find it profitable to supply liquidity with limit orders (compared with their outside option). Informed traders profit from both their information and liquidity provision.
  - When the spread is narrow, informed traders hide among the impatient uninformed traders to profit from their information.
- When testing this prediction, it should always possible to define two (or more) categories of spread size based on the historical distribution. Limit orders should carry more information when the spread is large, and market orders should carry more information form the spread is small.

Insight: How do Uninformed Traders Learn about the Asset Value?

- In each state of the book, the uninformed traders know if the informed traders use market or limit orders.
  - It is not the order choice of the informed traders that is driving the learning process.
- Any difference between the expected frequency and actual frequency of orders that could come from the informed traders provides information and leads to price adjustment.
- Updating the uninformed traders' beliefs leads to:
  - Changing the expectation of the asset value, v_t.
  - Changing their inference about the arrival rates (e.g., $\lambda_{t,LB} = \lambda_{bp} + q_t \lambda_I$).
- This means that not just the level of prices change, but also the distance between limit order prices.
  - The magnitude of the price revisions increases with the dispersion of the asset's value.

Insight: Repricing Limit Orders

- Without asymmetric information, once a limit order is submitted at a certain price, it rests in the book until it executes. With asymmetric information, every new order potentially brings about a revision in the uninformed traders' beliefs about the true value and a repricing of all limit orders: cancellations and resubmission at different prices.
- In traditional sequential trade models, only the best bid and ask prices (the dealer's quote) get updated. In our model, the impact of information results in price revisions of all limit orders.
- Our model suggests that both the frequency of cancellations and resubmissions and changes in the gaps between resting orders in response to order flow could be used empirically to help identify the presence of informed trading in limit order markets
- Empirically, we have seen a large number of limit order cancellations and resubmissions (sequences of limit orders that are revised). E.g., Hasbrouck and Saar (2013)'s strategic runs.
- Some people say that the frequent cancellations and resubmissions are evidence of nefarious HFT strategies. Others disagree (e.g., Baruch and Glosten (2013)).
- We provide another rationale for frequent cancellations and resubmissions of limit orders: it is the equilibrium response of limit order traders to the presence of informed traders.

Insight: “Regret-Free” Prices

- Our model can accommodate multiple conventions with respect to the timeline of updating beliefs.
- Convention 1: As an order arrives, beliefs are updated and then the order executes at a price that reflects its information content.
  - Prices at which trades execute are “regret-free”. In the spirit of van Kervel (2015)’s HFTs?
- Convention 2: Arriving order executes and only then limit order prices adjust to reflect its information content.
  - Consistent with auto-execution in U.S. markets.
- Can one implement the GMEO ex-ante “regret-free” prices in limit order books?
  - Which orders would you condition on, only market orders or limit orders as well?
  - Which prices should reflect this information, just the top prices or all limit orders in the book?
  - Traders learn from the time between orders. But with Poisson arrival, one cannot learn from the passage of time until the order actually arrives. So, ex-ante “regret-free” prices are impossible with Poisson arrival of traders!

Conclusions

- Same:
  - Order flow brings information to the market.
  - Uninformed traders incur execution costs when submitting market orders.
- Different:
  - Transaction costs reflect both expectations about the true value and the compensation that needs to be paid for liquidity provision.
  - Informed traders use both market and limit orders depending on the profitability of liquidity provision (or the size of the spread).
  - Not just the top of the book but all limit orders are repriced (cancelled and resubmitted) as traders learn information from the order flow.
- We hope that having a limit order model under asymmetric information that can be solved analytically would enable researchers to investigate many interesting questions.

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