Is 24/7 Trading Better?

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ABSTRACT

In a dynamic model of large traders who manage inventory risk, we show that a daily market closure coordinates liquidity. Some length of closure is welfare-improving relative to 24/7 trade, as the coordination of liquidity improves allocative efficiency, fully offsetting the costs of the closure. A long closure is optimal for traders in small markets, while traders in large markets would benefit from extending trading hours to near 24/7. A calibration of our model to several large equity exchanges that have proposed extending trading hours suggests that implementing such proposals would benefit traders.

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I. Introduction

Trading hours have historically aligned with the conventional workday due to the necessity of human involvement in both the submission and execution of trades. However, technological advancements have significantly reduced the need for human involvement, enabling many markets—such as futures, foreign exchange, and cryptocurrencies—to operate nearly continuously, often closing for only brief maintenance windows. Furthermore, the increased globalization of firms and the financial sector has generated new demand from market participants to respond to firm-relevant news as it emerges around the clock, often outside the firm's domestic trading hours.[1] In response, *some* major equity exchanges will soon extend their trading hours beyond the traditional 6.5-hour window, moving towards 23-hour trading days.[2] We analyze how changes in trading hours affect market liquidity and trader welfare.

We study a dynamic model of large traders managing risky inventory positions of a traded asset who rationally anticipate how their orders affect prices. Gains from trade are a result of both inventory cost sharing and reallocation across agents with stochastic private values. Traders optimally balance the benefits of eliminating undesired inventory against the costs of incurring price impact. We quantify the allocative efficiency of a market in equilibria of two market designs: one with a daily closure for a fixed fraction of the day and another with 24/7 trading. A daily closure is costly because it eliminates traders' ability to manage their inventory when the market is closed, leading traders to arrive at the start of the next day in positions that may be far from desirable.

Is there any benefit to a daily market closure? If there is a closure, traders rationally anticipate being unable to directly manage their inventory positions during the closure,

[1] Alternative trading systems (ATSs) have emerged to meet this demand, facilitating trading for certain exchange-traded products from 8:00 PM to 4:00 AM Eastern Standard Time. Eaton et al. (2025) document that 80% of the volume during these hours originates from the Asia-Pacific region.

[2] For example, 24X received SEC approval in November 2024 to launch the first registered 23/7 U.S. equity exchange. We provided a comment letter to the SEC explaining the implications of our work for 24X's proposal. 24X and the SEC cited our comment in a response letter and findings notice, respectively. The New York Stock Exchange polled market participants about 24/7 trading in April 2024 and is moving their Arca exchange's trading hours to 22/5. The Nasdaq and CBOE EDGX equities exchanges are similarly extending their trading sessions to 24/5. Robinhood, Charles Schwab, and Interactive Brokers already offer 24/5 access to selected equities and ETFs through ATSs such as Blue Ocean and EOS, respectively. However, the other ten U.S.-registered equity exchanges and popular international exchanges, such as the London and Tokyo Stock Exchanges, currently do not have public plans to extend trading hours.

which incentivizes them to further incur price impact by trading more aggressively towards a desirable position at the end of the trading day. In turn, this aggressive trading increases liquidity at the end of the day, which lowers the cost of trading and further incentivizes aggressive trading at the market closure. Therefore, liquidity is coordinated, and "liquidity begets liquidity," resulting in very low price impact and very efficient trade at the close.

Aware that liquidity will be coordinated in the final trading session of the day, traders have a strategic incentive to delay trade until the price impact is low in the last session of the day. This incentive to postpone trade within the day can be sufficiently large that there is an endogenous no-trade period in the sessions just preceding the final trading session. The incentive to postpone trade is largest in markets with few traders, where liquidity is spread relatively thin, and in markets where the payment of the asset's liquidating dividend is unlikely to occur before the next trading opportunity, making the cost of postponing trade relatively small. To summarize, although a daily closure has a natural cost by restricting traders' ability to respond to shocks, it has the benefit of coordinating trade at the closure. That benefit is also partially offset by the socially costly strategic delay of trade within the day. This strategic delay is consistent with empirical evidence that trade at closing auctions is highly concentrated, potentially at the expense of preceding sessions (e.g., AMF (2019)). Bogousslavsky and Muravyev (2023) find that their measure of illiquidity is seven times higher between 3:30 and 3:45 than between 4:00 and the closing auction.

The mechanisms of the model with closure are summarized through the behavior of intraday trade volume. We decompose traded quantities into two components that vary over time: a component that determines the gap a trader faces between their current and desired inventory, and a component that determines how aggressively a trader trades to eliminate the gap. Trade aggressiveness in a given session, the second component of trade, is increasing in liquidity. At the start of the day, traders face large gaps between their current and desired inventory levels, as shocks to their desired inventory position occur during the closure that traders are unable to respond to. This generates a large volume at the start of the day despite relatively low trade aggressiveness. At the end of the day, traders trade very aggressively to close any gap that remains. So, even though trade earlier in the day shrinks the gap between current and desired inventory, this aggressive trade at the liquid closing session results in large volume. In the middle of the day, the gaps between traders' desired and current inventories are not particularly large, and trade is not particularly aggressive, resulting in low volume compared to other parts of the day. Thus, as in the data (e.g., Chan et al. (1996), Jain and Joh (1988)), intraday volume exhibits a U-shaped pattern.

When trade is 24/7, there is no equilibrium in which traders coordinate trade. Since traders rationally anticipate how their demand affects prices and future inventory positions, they break up their orders over time to minimize execution costs, leading to socially inefficient excess inventory costs (Du and Zhu, 2017b, Rostek and Weretka, 2015, Vayanos, 1999). Liquidity is spread out, and price impact further increases, further incentivizing traders to break up their orders. With 24/7 trade, liquidity is spread thinly throughout the trading day. A market closure can potentially benefit traders by coordinating liquidity.

Next, we quantify trader welfare in various market designs. We show that there is always a length of closure that is better than having trade 24/7. The optimal closure may be short. We find the optimal length of closure is longer in smaller markets, that is, markets where the number of traders and the rate of shocks to private values are small. In markets with a large number of traders, liquidity is already substantial, minimizing the relative benefits of coordinating trade. In markets in which shocks to private valuations are frequent, the costs of restricting traders' ability to respond to these shocks are high, implying a short closure is optimal.

We calibrate our model to four different equity exchanges—NYSE, Nasdaq, CBOE EDGX, and NYSE Arca—to assess the policy implications of likely changes to the current U.S. equity market structure.[3] We choose these four exchanges because the NYSE is the largest registered U.S. equity exchange, and the Nasdaq, CBOE EDGX, and NYSE Arca have announced plans to extend to 24/5, 24/5, and 22/5 trading days, respectively. We calibrate the model to match the model-implied intraday volume to the empirical intraday volume. The calibration suggests that, for the exchanges we consider, the proposed changes in trading hours will benefit traders, and a very short closure of 2 to 7 minutes is optimal. Our results suggest that the NYSE should follow suit and extend its trading hours, as should other large equity exchanges, such as the London and Tokyo stock exchanges, which currently have no plans to do so. The calibrated welfare gains relative to the current market structure are similar across counterfactuals with 23/7 trade, 24/7 trade, and the optimal closure.

Our main results are robust to allowing traders to observe noisy private signals about fundamental asset values. Heterogeneity tends to reduce the aggressiveness of trade overall,

[3] Although we focus on equity markets, the theoretical framework is applicable to other asset classes.

as it introduces a price impact resulting from adverse selection. Yet, closure still coordinates liquidity, improving welfare by allowing traders to trade very aggressively at the end of the day with minimal price impact, especially in markets with few traders or with infrequent shocks to private and fundamental values.

The purpose of this paper is to evaluate the merits of changing trading hours in a framework that accounts for price impact, the first-order concern for large traders. Indeed, Frazzini et al. (2018) empirically documents that price impact is the only first-order trading cost for a large asset manager. In this respect, this paper follows a series of papers, including Chen and Duffie (2021), Antill and Duffie (2020) and Du and Zhu (2017b), that evaluate market structures in frameworks that consider price impact and the strategic incentives of traders. Nevertheless, there are additional factors that should be considered when policymakers evaluate the merits of extending trading hours. These include the incentives of exchanges, particularly when markets are fragmented, the incentives of firms, the effects of trading hours on the efficiency of closing prices, implications for international participation in financial markets, regulatory concerns, and implications for retail traders. We discuss each of these factors in detail in Section VII. Given exchanges propose changes to trading hours, their incentives are particularly important. Although a full analysis of their fee contracting and competitive incentives in fragmented markets is beyond the scope of this paper, we do analyze the implications of changes in market structure for volume, which is a primary source of revenue for exchanges. In particular, in large markets, we find that extending trading hours would increase daily volume in the model, suggesting that their incentives are largely in line with those of traders, as suggested by our calibrated welfare results.

Literature Review

There is extensive literature empirically documenting intraday and overnight patterns in financial markets.[4] A substantial literature theoretically explains these facts (Hong and Wang, 2000, Subrahmanyam, 1994, Foster and Viswanathan, 1993, Brock and Kleidon, 1992, Foster and Viswanathan, 1990, Admati and Pfleiderer, 1989, 1988). However, these studies treat the duration of the daily market closure as fixed. This paper differs by varying the

[4] For example, Bogousslavsky (2021), Hendershott et al. (2020), Lou et al. (2019), Branch and Ma (2012), Kelly and Clark (2011), Cliff et al. (2008), Branch and Ma (2006), Andersen and Bollerslev (1997), Chan et al. (1996), Amihud and Mendelson (1991), Stoll and Whaley (1990), Barclay et al. (1990), Harris (1989, 1988), Amihud and Mendelson (1987), Harris (1986), Fama (1965).

length of closure and analyzing welfare in a dynamic setting with endogenous price impact, a first-order concern for large traders. Nonetheless, some of these studies do offer mechanisms related to ours. In a competitive setting, Hong and Wang (2000) takes an asset pricing perspective and studies the implications of restricting overnight trade; we take a market design perspective and show that under imperfect competition, restricting overnight trade can actually enhance allocative efficiency, a novel and central result. Admati and Pfleiderer (1988) and Foster and Viswanathan (1990) find that noise traders may concentrate trade to mitigate adverse selection, although they do not study intraday patterns of trade concentration; we show that even homogeneous traders will strategically cluster trades before a daily closure.

This paper also contributes to the literature on how common financial market structures interact with strategic trading and the implications for the allocative efficiency of the market (Rostek and Yoon, 2025). Chen and Duffie (2021), Malamud and Rostek (2017), and Kawakami (2017) study market fragmentation. Fuchs and Skrzypacz (2019), Du and Zhu (2017b) and Vayanos (1999) study trading frequency. Antill and Duffie (2020), Duffie and Zhu (2017), and Blonien (2024) examine the addition of a trading session at a fixed price. Chen et al. (2024), Kodres and O'Brien (1994), Subrahmanyam (1994), and Greenwald and Stein (1991) study circuit breakers. Fuchs and Skrzypacz (2015) study government market freezes in a dynamic adverse selection model. Apart from being the endogenous outcome of adverse price movements, circuit breakers do share conceptual similarities with daily closures. Although none of these papers study the implications of daily market closures for both allocative efficiency and liquidity in a dynamic model.

Bid shading, or the strategic delay of trade, is a standard result in dynamic models with price impact and strategic trade. In studies such as Antill and Duffie (2020), Du and Zhu (2017b), and Vayanos (1999), strategic delay is a direct response to a change in market structure. In this paper, the strategic delay before the close is a strategic response to the endogenous coordinated trade at the end of the trading day. The coordinated trade at the end of the day is, in some sense, the opposite of strategic delay, as traders rush to the market in anticipation of worsening investment opportunities overnight. The fact that traders can exhibit oscillatory-type strategic delay, and that it can be so strong as to preclude trade in the periods just prior to the close, is theoretically novel, and an illustration of the complex patterns that can arise in non-stationary dynamic trade.[5]

deHaan and Glover (2024) is a recent paper whose focus is on the empirical portfolio performance of retail traders as a function of trading hours. We do not directly model retail traders. Over 90% of retail marketable orders are internalized by wholesalers off-exchange (Gensler, 2022), suggesting that changes in exchange hours will primarily affect retail traders through their effect on wholesalers. If the allocative efficiency gains in our model are transmitted to retail traders through better pricing or execution, our calibration suggests that extended trading hours would be beneficial for retail traders.

The presence of market closures is closely linked to the existence of closing auctions, whose characteristics have been of recent interest. Bogousslavsky and Muravyev (2023), Jegadeesh and Wu (2022), and Hu and Murphy (2025) empirically study liquidity and price efficiency around the NYSE and Nasdaq closing auctions. The percentage of daily volume transacted in these special sessions has reached an all-time high in recent years (Bogousslavsky and Muravyev, 2023), consistent with our model, which generates substantial volume near the opening and closing. Our model predicts that if trading hours are extended, trading volume will be less concentrated at the opening and closing sessions. The Autorité des Marchés Financiers (AMF, 2019) has warned that concentration at the close could harm price efficiency and liquidity beforehand. We find that although liquidity does deteriorate as traders delay for the closing auction, the resulting social costs can be outweighed by the coordination benefits of closure.

The paper proceeds as follows. Section II defines the model. Section III defines and solves for the equilibrium and builds intuition for how the traders optimally trade with and without a market closure. Section IV quantifies welfare. Section V calibrates the model to several equity exchanges. Section VI extends the model to allow for heterogeneous signals about a common dividend. Section VII discusses additional factors beyond price impact that should be considered when evaluating the merits of extending trading hours. Section VIII concludes. The Appendices provide technical details and proofs.

[5] Rostek and Weretka (2015) also has non-stationary market characteristics in a slightly different equilibrium concept. In their setting, price impact is non-stationary and depends on the timing of information about the dividend throughout the session, although equilibrium allocations are stationary functions of state variables. In our setting, the end of the trading day coordinates and improves liquidity and increases the trade aggressiveness embedded in demand schedules.

II. The Model

This section introduces a model of strategic trading under imperfect competition with periodic market closures. Time is continuous and goes from 0 to ∞. We set a unit of clock time to be 24 hours. Each 24 hour period is divided into K evenly spaced subperiods of length $h := \frac{1}{K}$. Trade occurs the first $T+1$ periods, and no trade is permitted in the last Δ periods. We refer to the fraction of the 24 hours when trade can occur as "day," and the remaining fraction is referred to as "night."

Let us illustrate this setup in the first day, where clock time t is in $[0,1)$. Trade occurs at times $0, h, \ldots, Th$, and the night spans times $(Th, 1)$, which includes times $(T+1)h, (T+2)h, \ldots, (T+\Delta)h$. Note $(T+\Delta)h = 1-h$. At time 1, the next day starts, and the timing repeats.

There are $N \geq 3$ risk-neutral traders who trade a divisible asset. Traders want to hold the asset because it pays a liquidating dividend of v per unit of inventory held. The time to liquidation is exponentially distributed, denoted $\mathcal{T} \sim Exp(r)$, so that the expected time until liquidation is $\frac{1}{r}$. Each trader is endowed with some portion of the asset, referred to as the trader's initial inventory. In addition to differing endowments, traders have private values that motivate trade (Harris and Raviv, 1993). We assume a private value of $w^i_{\mathcal{T}}$ per unit of the asset is realized upon liquidation. Thus, the total value of the asset at liquidation is $v + w^i_{\mathcal{T}}$. The private value, w^i_t, is a continuous-time random walk with normally distributed zero-mean increments with standard deviation σ that arrive at a constant rate λ. These shocks are independent across time and traders and independent of all other shocks in the model. Shocks to private values induce continued gains from trade over time. These shocks can be motivated by risk management considerations or shocks to preferences. They simply represent a reduced-form motive for trade, whether due to behavioral or rational reasons.

Each trading session is modeled as a uniform-price double auction. Each trader i submits a demand schedule $D^i : \mathbb{R} \to \mathbb{R}$ that is a mapping of price to demand, $p \mapsto D^i(p)$. The market-clearing price, p^*_t, is the price that sets net demand to be zero,

$$\sum_{i=1}^{N} D^i(p^*_t) = 0. \tag{1}$$

Each trader pays the equilibrium price, p^*_t, times the amount of the asset they were allocated, $D^i(p^*_t)$. If $D^i(p^*_t) < 0$, then trader i receives the price times the amount of the asset they sell.

The modeling of trade as an auction, as opposed to a limit-order book, provides tractability while maintaining the important economic mechanism of price impact from trade.

Traders in the model dynamically manage inventory positions. Define trader i's inventory of the asset at time t to be z_t^i, and the average aggregate inventory, $\bar{Z} := \frac{1}{N}\sum_{i=1}^{N} z_t^i$, is a constant. After trade at time t, trader i's inventory moves to $z_t^i + D^i(p_t^*)$. In addition to trading due to heterogeneous private values of the asset, traders also trade to manage inventory costs. In particular, we assume traders incur a holding cost per unit of time of $\gamma \times (z_t^i)^2$.Chen and Duffie (2021), Antill and Duffie (2020), Duffie and Zhu (2017), Du and Zhu (2017b), Sannikov and Skrzypacz (2016), Rostek and Weretka (2012), Vives (2011), Blonien (2024) and Chen (2022) all use a similar quadratic holding cost. This cost can be interpreted as representing inventory costs or collateral requirements. More generally, including these exogenous inventory costs is a reduced-form approach to modeling incentives to risk share.[6]

Since traders can only manage inventory through trade during the day, and private values can be shocked during the day or overnight, the restrictions that market closures impose have obvious costs. If a shock to private values arrives overnight, traders will arrive at the start of the next day at positions that are suboptimal. In the model, traders trade off maintaining suboptimal inventory positions against price impact costs. Therefore, they trade slowly toward their desired inventory position, potentially heightening the costs of a temporary closure. This paper's goal is to study the costs and benefits of daily market closures through the organization of trade they induce.

Now, let us define the traders' optimization problem. In the following sections, we will study equilibria that are periodic, with a period of one day. Therefore, to ease the exposition, we simply focus on time $t \in [0, 1)$ and note that expressions at any other time are analogous. Recall that trade during the first day occurs at times $0, h, \ldots, Th$. For $t = kh$ in any of these periods apart from the last, denote any trader's value function V_k. The value function is a function of current inventory position z^i, current private value w^i, and average aggregate private value $\bar{W} = \frac{1}{N}\sum_{i=1}^{N} w^i$, and satisfies the following Bellman equation:

[6] Having described the model, it is worth noting slightly different assumptions— continuously paid liquidating dividends, repeatedly paid dividends, private value shocks at pre-determined arrival times, correlated private value shocks, private signals about a risky common value v (see Section VI), and time-varying deterministic inventory costs or private value shocks—do not substantively change the mechanisms of the model.

$$V_k(z^i, w^i, \bar{W}) = \max_{D^i} \Bigg\{ -\underbrace{D^i p^*_{kh}}_{\substack{\text{cost of}\\ \text{trade}}} + \underbrace{(1 - e^{-rh})}_{\substack{\text{prob. of}\\ \text{liquidation}}} \underbrace{(z^i + D^i)(v + w^i)}_{\text{liquidation value}}$$

$$- \underbrace{\frac{(1 - e^{-rh})}{r}}_{\substack{\text{expected length}\\ \text{of flow cost}}} \underbrace{\frac{\gamma}{2}(z^i + D^i)^2}_{\substack{\text{inventory}\\ \text{flow cost}}} + \underbrace{e^{-rh}}_{\substack{\text{prob. of no}\\ \text{liquidation}}} \underbrace{E_{kh} V_{k+1}(z^i + D^i, w^i_{(k+1)h}, \bar{W}_{(k+1)h})}_{\text{expected future value}} \Bigg\}. \quad (2)$$

The maximum is over demand schedules, not simply realized demands. The first term corresponds to the cost (allocated quantity times the market-clearing price) of trade incurred in the double auction at time kh. The next term corresponds to the expected payoff if the asset liquidates before the next session times the probability it liquidates before the next session. The third term is the expected holding cost before the next session, which incorporates the probability that the asset might liquidate, after which there is no more holding cost. The last term is the next period's continuation value, assuming the asset does not liquidate before then, times the probability the asset does not liquidate before the next period. As we will show, prices reveal the average private value $\bar{W}$ in equilibrium. Therefore, the value function is a function of $\bar{W}$ insofar as it affects future prices and realized demands and, thus, utility. In the last trading period of the day, that is the $(T+1)^{th}$ trading session at clock-time Th, the Bellman equation is modified to the following:

$$V_T(z^i, w^i, \bar{W}) = \max_{D^i} \Bigg\{ -\underbrace{D^i p^*_{Th}}_{\substack{\text{cost of}\\ \text{trade}}} + \underbrace{(1 - e^{-rh(1+\Delta)})}_{\substack{\text{prob. of}\\ \text{liquidation}}} \underbrace{(z^i + D^i)(v + w^i)}_{\text{liquidation value}}$$

$$- \underbrace{\frac{(1 - e^{-rh(1+\Delta)})}{r}}_{\substack{\text{expected length}\\ \text{of flow cost}}} \underbrace{\frac{\gamma}{2}(z^i + D^i)^2}_{\substack{\text{inventory}\\ \text{flow cost}}} + \underbrace{e^{-rh(1+\Delta)}}_{\substack{\text{prob. of no}\\ \text{liquidation}}} \underbrace{E_{Th} V_0(z^i + D^i, w^i_1, \bar{W}_1)}_{\text{expected future value}} \Bigg\}. \quad (3)$$

The terms are modified to reflect the increased likelihood that the asset liquidates before the next trading session, as there are $h(1 + \Delta)$ units of clock time between trade instead of h.

III. Equilibrium

Section IIIA studies the equilibrium of the model of strategic trading under imperfect competition with periodic market closures. Section IIIB studies a version of the model without market closure that is a special case of the model studied in Du and Zhu (2017b). Sections IIIC through IIIE study results that describe the solution of the model.

A. *Equilibrium with a Daily Closure*

Prior studies of uniform-price double auctions (e.g., Antill and Duffie (2020), Du and Zhu (2017b), Vayanos (1999)) frequently consider equilibria that are symmetric, linear, and stationary. That is, the equilibrium demand schedules of each trader are the same linear combination across time of price and other relevant state variables. In our model with daily market closures, such an equilibrium will generally not exist. The trading problem that every trader faces is not ex-ante identical at each trading session, as the opportunity set changes throughout the day, precluding the existence of stationary equilibria. For instance, as the closure approaches, traders will behave differently since the inability to manage inventory overnight presents a substantial change to their opportunity set.

Therefore, we focus on symmetric, linear, and daily-periodic demand schedules. For example, in equilibrium, all demand schedules submitted at 9:30 AM will be the same function every day, but all traders may use a different demand schedule at 10:00 AM than they did at 9:30 AM. Thus, the equilibria we consider are stationary across days but not within the same day. Concretely, we conjecture that the equilibrium demand schedule at trading session $k \in \{0, \ldots, T\}$ is of the following form:

$$D_k^i(z^i, w^i, p) = a_k + b_k p + c_k z^i + f_k w^i, \tag{4}$$

and $b_k \leq 0$. By market clearing, in equilibrium, trader i will face the residual supply curve of the other $N - 1$ traders and effectively choose a price and quantity pair.

In addition to allowing the submission of periodic, not constant, demand schedules, we also differ from prior literature by allowing investors to submit demand schedules whose slopes b_k are 0. If trader i chooses demand quantity d^i, then by market clearing, the price must solve $d^i + \sum_{j \neq i}(a_k + b_k p + c_k z^j + f_k w^j) = 0$. If $b_k = 0$, there is generally no market-clearing price in a symmetric, linear equilibrium, unless the submitted demand schedules are uniformly equal to 0, in which case any price clears the market. In other words, allowing b_k to equal 0 is akin to allowing the traders to abstain from trade. Intuitively, if other traders submit demand schedules equal to zero, it is equilibrium behavior for trader i to submit a demand schedule equal to zero, since trader i will be allocated zero regardless of the price.

Now let us consider the case $b_k < 0$. This case corresponds to periods k with non-zero trade. If trader i chooses demand quantity d^i, then by market clearing, the price must solve

$d^i + \sum_{j \neq i}(a_k + b_k p + c_k z^j + f_k w^j) = 0$. Therefore, the market-clearing price is

$$\Phi_k(d^i, z^i, W^{-i}) := p = -\frac{1}{b_k(N-1)}\big(d^i + (N-1)a_k + c_k(N\bar{Z} - z^i) + f_k W^{-i}\big), \qquad (5)$$

where $W^{-i} = \sum_{j \neq i} w^j$. Traders are strategic, and thus, they rationally anticipate and internalize how their demand affects prices due to imperfect competition. As price impact itself is only a wealth transfer between traders, it is the strategic effects of avoiding price impact that can be socially costly by reducing allocative efficiency.

A symmetric (Markov perfect) equilibrium of the above stochastic game is defined by the sequences $(a_k)_{k=0}^T$, $(b_k)_{k=0}^T$, $(c_k)_{k=0}^T$ and $(f_k)_{k=0}^T$. Equilibrium requires that if trader i conjectures the other $N-1$ traders use the demand schedule from equation (4), trader i's best response is to submit the same demand schedule, and the market clears. It is important to note that we do not assume that trader i must play the conjectured form of the demand schedule, but it will be their best response to do so, contingent on others submitting linear, symmetric, and daily-periodic demand schedules.

There are multiple equilibria when traders can submit zero demand schedules in any period. If all other traders submit zero demands, it is equilibrium behavior for any trader to do the same, as market clearing implies that residual demand, and thus their own equilibrium demand, is 0. Thus, in principle, traders can abstain from trade in any combination of periods during the trading day. In particular, two general classes of equilibria are possible: one in which no-trade periods occur only when there is no symmetric and linear non-zero trade equilibrium that period, and another class in which there are periods in which investors do not trade in some periods, even though there is a symmetric and linear non-zero trade equilibrium in at least one of the no-trade periods. The former class of equilibria is unique, while the latter can greatly increase the number of possible equilibria. To restrain the number of equilibria, we require that equilibria also satisfy a trembling-hand refinement. Specifically, from the perspective of trader i, assume the other $N-1$ traders jointly tremble between two possible equilibria, playing one with probability $1-q$ and the other with probability q. Then, we consider the limiting behavior of investor i as $q \to 0$. In particular, for the equilibrium whose probability of being played converges to 1 to survive the refinement, we require that trader i's optimal demand schedule converges to its equilibrium demand at any date. This refinement rules out fragile equilibria by selecting equilibria that are robust to potential deviations from the equilibrium path. Moreover, it selects equilibria from the class for which no-trade periods occur only when there is no symmetric and linear non-zero trade equilibrium that period. To summarize, the equilibrium of the demand submission game that we study is defined as follows:

DEFINITION 1: *Equilibrium of the demand submission game is described by the sequences* $(a_k)_{k=0}^T$, $(b_k)_{k=0}^T$, $(c_k)_{k=0}^T$ *and* $(f_k)_{k=0}^T$. *In each period in the trading day, equilibrium requires that if trader* i *conjectures the other* $N-1$ *traders submit the demand schedule*

$$D_k^i(z^i, w^i, p) = a_k + b_k p + c_k z^i + f_k w^i,$$

where $b_k \leq 0$, *trader* i*'s best response is to submit the same demand schedule, and the market clears. Moreover, the equilibrium must satisfy the trembling-hand refinement.*

We show in Appendix A that an equilibrium exists, is unique, and is characterized by Proposition 1.

PROPOSITION 1: *There is a unique equilibrium of the demand submission game. Moreover, the equilibrium has the following properties:*

1. *There is always non-zero trade in the last period of each day, period* T.
2. *If there is at least one period with trade prior to period* T, *then it consists of a sequence of contiguous periods with non-zero trade followed by a contiguous no-trade period, either of which may be of length zero.*
3. *In periods with non-zero trade, the equilibrium quantity traded takes the form*

$$D_k^i(p_{kh}^*) = c_k\left(z_{kh}^i - \left(\frac{r}{\gamma}(w_{kh}^i - \bar{W}_{kh}) + \bar{Z}\right)\right), \tag{6}$$

   where $k \in \{0, \ldots, T\}$, *for* $c_k \in [-1, 0)$ *characterized in Appendix A. The equilibrium market-clearing price is*

$$p_{kh}^* = v + \bar{W}_{kh} - \frac{\gamma}{r}\bar{Z}. \tag{7}$$

4. *Let* $\bar{c}$ *denote the equilibrium value of* c_k *if there is no market closure, as given below in Proposition 2. In two consecutive periods of trade* $k, k+1$, *if* $c_k > \bar{c}$, *then* $c_{k+1} < \bar{c}$. *Similarly, if* $c_k < \bar{c}$, *then* $c_{k+1} > \bar{c}$. *An analogous pattern applies to* $1/b_k$, *which determines price impact.*

Even though a unique equilibrium exists, non-zero trade does not necessarily occur every period during the day. Equivalently, there may be periods in which $b_k = 0$ in equilibrium.

The equilibrium is unique even though trade can equal 0 in a given period because in periods without trade, there is no submission of non-zero demand curves that form an equilibrium; in equilibrium, traders abstain from trade if and only if there is no equilibrium with non-zero trade in a given period. Proposition 1 shows that there will be a contiguous period of non-zero trade, followed by a contiguous period without trade, followed by a final period with trade. The overnight closure of length Δ then follows the final period of trade. As stated in the proposition, it is worth bearing in mind that the contiguous periods of trade or of no trade may be of length 0, but there is always non-zero trade at the close.

Let us discuss these results. We will begin by discussing the allocations in the model, described in property 3 of Proposition 1. Then, we will discuss the strategic incentives in the model, summarized in property 4, and what these strategic incentives imply regarding the patterns of trade throughout the trading day, summarized by properties 1 and 2.

First, let us look at the functional form of the allocation, $c_k(z^i - (\frac{r}{\gamma}(w^i - \bar{W}) + \bar{Z}))$. The allocation is the current inventory net of a measure of desired inventory, which we define as $\tilde{z}^i := \frac{r}{\gamma}(w^i - \bar{W}) + \bar{Z}$, multiplied by c_k. $\tilde{z}^i$ is the inventory position a trader would reach each period after trade if the market were competitive. We refer to $\frac{r}{\gamma}(w^i - \bar{W}) + \bar{Z}$ as desired inventory because if $z^i = \frac{r}{\gamma}(w^i - \bar{W}) + \bar{Z}$ for every trader, then there is no more trade in equilibrium. Consider the post-trade inventory position,

$$z^i_{k+1} = z^i_k + D^i_k(p^*_k) = (1 + c_k)z^i_k - c_k\tilde{z}^i_k. \tag{8}$$

Recalling that c_k lies in $[-1, 0)$ when there is trade, c_k measures trade aggressiveness as it is the fraction of trader i's new inventory position that is made up of their old inventory position, and the remaining fraction is the desired inventory position. Subtracting $\tilde{z}^i_k$ from both sides of Equation 8, the gap between the next period's inventory and the desired inventory is

$$z^i_{k+1} - \tilde{z}^i_k = (1 + c_k)\big(z^i_k - \tilde{z}^i_k\big). \tag{9}$$

As c_k approaches -1, which is its value under perfect competition, this gap approaches zero, and the allocation of the asset becomes more efficient.

We will see numerically in the remaining sections that the coefficient c_k in the equilibrium allocation is negative and largest in absolute value in the last session of the trading day. As the end of the day approaches, traders are aware that they will soon lose the opportunity to manage random shocks in their desired inventory positions through trade. They all, therefore, have the incentive to enter the closure in a desirable inventory position. As a result, traders are more willing to incur price impact and temporary trading costs toward the end of the trading day. The old adage of "liquidity begets liquidity" comes into effect; liquidity improves due to the symmetric fear of suboptimal inventory positions being exacerbated overnight, making it even cheaper to trade more aggressively now, further encouraging aggressive trading.

This incentive to enter overnight in a good position is strongest in the final period of trade. In fact, by backward induction, traders know that trading costs will be low in the final period. Therefore, traders have an incentive to postpone trading until then, thereby reducing liquidity in the penultimate period. This explains property 4 of the equilibrium, which formalizes the strategic incentives in the model. Essentially, if trade is aggressive in the next period, trade is less aggressive in this period, as traders postpone to the next period when the price impact is lower. Similarly, if trade is less aggressive in the next period, trade will be more aggressive in this period. Thus, trade has some oscillatory properties. In our numerical examples, the oscillations in c_k decay quickly as traders move backward in time from the final trading sessions.

The incentives to postpone trade are smallest when N, the market size, is large or when rh, the per-period discount rate, is large. When there are many traders, price impact is generally small, implying that the benefits of a liquid final period of trade are muted. If the per-period discount rate, rh, is large, the costs of delaying trade are large as the asset is more likely to liquidate before the next trading opportunity. Formally, if the incentives to postpone trade are not large, we arrive at an equilibrium with trade every period:

COROLLARY 1: *If* $(N-1)(1-e^{-rh}) > 1$*, the unique equilibrium has non-zero trade in all periods,* $0, \ldots, T$*.*

The condition $(N-1)(1-e^{-rh}) > 1$ is a sufficient condition that describes how large N and rh must be in order for trade to occur every period. If the incentives to postpone trade are sufficiently strong, the equilibrium with trade every period breaks down, and there is at least one period of no trade leading up to the closing session. Empirically, an analog of this result is the fact that in markets with closing auctions, liquidity prior to the closing auction is relatively thin, as trade is delayed due to the coordination in the closing auction (Bogousslavsky and Muravyev (2023), AMF (2019)). We study the length of the period of no trade in the model in Section IIIC.

Before moving on to analyze the model in more detail, we note that there is a continuous trade version of the model, which we will make use of when analyzing welfare. In this model, trade occurs at a rate in a continuous sequence of uniform-price double auctions for the first $1-\Delta-\epsilon$ units of the day, there is a no-trade period for the next endogenous length ϵ units of time, and a closing auction occurs at time $1-\Delta$. We slightly abuse notation by defining $\Delta \in [0,1)$ to be the fraction of the day that the market is closed in the continuous trade model, whereas it is the number of periods the market is closed in the discrete trade model. The derivation of this continuous trade equilibrium is in Internet Appendix IA.4, where we also show the convergence of the discrete trade model. In this version of the model, the length of the no-trade period can be determined analytically, with no parameter restrictions apart from $N \geq 3$. Moreover, prior to the no-trade period, demand schedules are stationary and thus do not depend on time and so do not oscillate.

It is worth highlighting some of the expressions in the continuous trade version of the model, as quantities such as c_k and b_k for the discrete trade model are provided in the Appendix but are not readily interpretable. In the continuous trade model, the length of the no-trade period ϵ is

$$\epsilon = \min\left\{1-\Delta, \frac{1}{r}\log\left(\frac{e^{-\Delta r}+(1-e^{-\Delta r})N}{e^{-\Delta r}+(1-e^{-\Delta r})(N-1)}\right)\right\}. \tag{10}$$

For $\epsilon < 1-\Delta$, the coefficient c_T in the demand function at the close, $1-\Delta$, is

$$c_T = -\frac{(N-2)(1-e^{-\Delta r})}{e^{-\Delta r}+(1-e^{-\Delta r})(N-1)}, \tag{11}$$

and $c_T = \frac{\gamma}{r} b_T$. It's straightforward to see that ϵ is increasing in Δ (as long as the minimum above does not bind) and decreasing in N, while both c_T and b_T become more negative as N and Δ increase. These comparative statics are analyzed in further detail in the discussion surrounding Figure 2 below.

B. *Equilibrium Without a Daily Closure – 24/7 Trading*

Let us briefly review the solution without market closure and then compare the two models. We make no other modifications to the model from the previous section other than setting $\Delta = 0$. Once again, we differ from most prior literature by conjecturing linear, symmetric, and periodic equilibria of the same form as Equation 4, and by allowing demand to be uniformly zero in any period. Periodicity again requires the demand schedules to be periodic functions of time with period 1.

We characterize the equilibrium in Proposition 2.

PROPOSITION 2: *When $\Delta = 0$, there exists a unique equilibrium. The equilibrium has the following properties:*

1. *The equilibrium quantity traded takes the form*
$$D_k^i(p_{kh}^*) = \bar{c}\bigg(z_{kh}^i - \big(\frac{r}{\gamma}(w_{kh}^i - \bar{W}_{kh}) + \bar{Z}\big)\bigg), \tag{12}$$
where $k \in \{0, \ldots, T\}$, and $\bar{c} \in [-1, 0]$ and is equal to
$$\bar{c} = \frac{-(N-1)(1-e^{-rh}) + \sqrt{(N-1)^2(1-e^{-rh})^2 + 4e^{-rh}}}{2e^{-rh}} - 1.$$
2. *The equilibrium market-clearing price is*
$$p_{kh}^* = v + \bar{W}_{kh} - \frac{\gamma}{r}\bar{Z}. \tag{13}$$

The equilibrium strategy played is time-invariant. Despite allowing the demand schedules submitted to be periodic across days, the unique equilibrium is constant across time, as in Du and Zhu (2017b). Additionally, despite allowing submitted demand curves to equal 0, no equilibrium with zero trade in a period satisfies the trembling-hand refinement. Thus, this equilibrium is a special case of Du and Zhu (2017b) in which there is no adverse selection. In the model with closure, trade is non-stationary throughout the day. Importantly, this non-stationarity leads to a coordination of liquidity towards the end of the day.

It is worth noting that prices are the same when trade is 24/7. In equilibrium, the first-order condition for optimal demand implies that the price has to equal the average marginal value of the asset. That is, $p_{kh}^* = \frac{1}{N}\sum_{i=1}^N \frac{\partial V_k}{\partial z^i}$. This average marginal value does not depend on price impact since price impact is a transfer across traders. It is only a function of the marginal benefit of holding the asset, which depends on the common and private values, and the marginal cost of holding the asset, which depends on γ.

C. Equilibrium Intuition

In this section, we compare the equilibrium in Proposition 1 with a market closure to the equilibrium in Proposition 2 with 24/7 trade. The introduction of an overnight closure, which lasts $h(1+\Delta)$ units of clock time, creates non-stationarity in the equilibrium demand

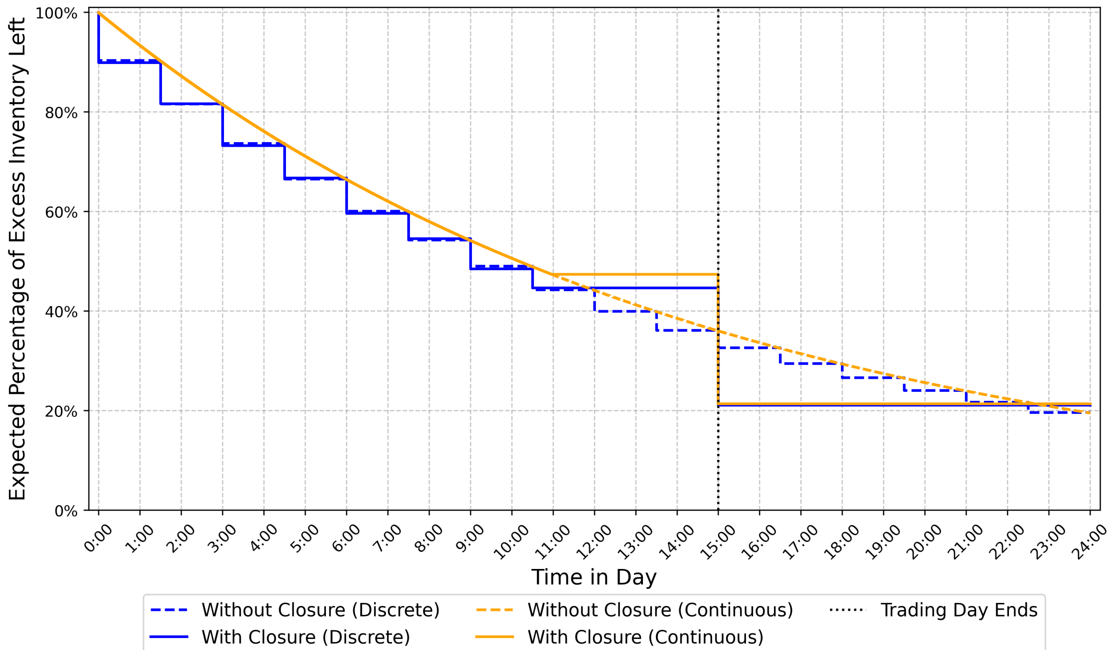


Figure 1. Trading Intensity Throughout the Day
This figure plots trading intensity for various regimes throughout the day. The y-axis is the expected percentage of the time 0 excess inventory left at time t in the day. The solid lines are market designs with a closure of 31.25% of the day from Proposition 1, and the dashed lines are market designs without a closure from Proposition 2. The colors map to the trading frequency of the market, with blue being $K = 16$ periods a day and orange being continuous trading. The vertical dotted line is when the market closes for trading for structures with closure. We use $N = 100$ and $r = 1/30$.

functions. In the 24/7 model, the aggressiveness of traders, c_k, is constant over time. In the model with a daily closure, trade aggressiveness has three distinct periods of behavior. Let us discuss this through the example displayed in Figure 1.

Figure 1 quantifies the aggressiveness of trade when there is a market closure, separately for discrete and continuous trade versions of the model. The y-axis is the percentage of excess inventory left relative to the start of the day for a given trader, assuming neither shocks to private values nor asset liquidation occur. Recall excess inventory is simply the difference between current inventory, z_t^i, and desired inventory, $\tilde{z}_t^i$, which is closed by $1 + c_k$ in trading session k. Mathematically, the y-axis is $\prod_{j=0}^{k}(1 + c_j)$, where k is the $(k+1)^{th}$

trading session of the day, which occurs at clock time kh.

When trade is 24/7, c is constant and between -1 and 0, and traders close $|c|$ percent of the excess inventory each period. When the trading frequency is higher, liquidity per trading session is lower, which increases price impact, which further reduces traders' willingness to trade. Du and Zhu (2017b) studies the tradeoff between this strategic cost and the ability to react to shocks more quickly by quantifying the optimal trading frequency in financial markets.

When there is a daily closure, the strategic incentives dramatically change the equilibrium trading patterns. Let us work backwards in time. Starting at the close, traders rationally anticipate that they will be stuck in an inventory position overnight, which will incur flow costs overnight irrespective of the shocks to their private values, and there is also some chance the asset will liquidate. Moreover, traders will not be able to react to shocks to private values that occur overnight, making excess inventory at the end of the day even less desirable. These risks increase traders' marginal willingness to incur additional price impact at the end of the day to avoid a worse inventory position at the start of the following day, which will take many trading sessions to correct due to price impact-induced bid shading. This incentive is present among all traders. As they all trade more aggressively, liquidity increases, which in turn decreases the price impact. Therefore, traders become even more aggressive, and this logic repeats. The closure helps traders coordinate their trades, which are otherwise broken into child orders when trading is 24/7. This can be seen in the plot by the large downward jump in the amount of excess inventory held immediately after the last trading session of the day. When trade is 24/7, the amount of excess inventory is larger overnight than when there is a daily closure, except toward the very end of the night.

Trade is very efficient at the close, and traders are rational and strategic. Therefore, in periods leading up to the closure, traders would like to delay trade in order to trade very cheaply at the close. This incentive to delay trade is so strong that, in the plotted example, there is no trade in the periods just preceding the close. Empirically, this is consistent with Bogousslavsky and Muravyev (2023), which finds that illiquidity is seven times higher between 3:30 and 3:45 than at the close.

In trading periods near the start of the trading day, undesired flow costs and liquidation risk throughout the day are sufficiently large that it is worth incurring some price impact to optimize positions, and there is non-zero trade. When trade is continuous, trade aggressive-

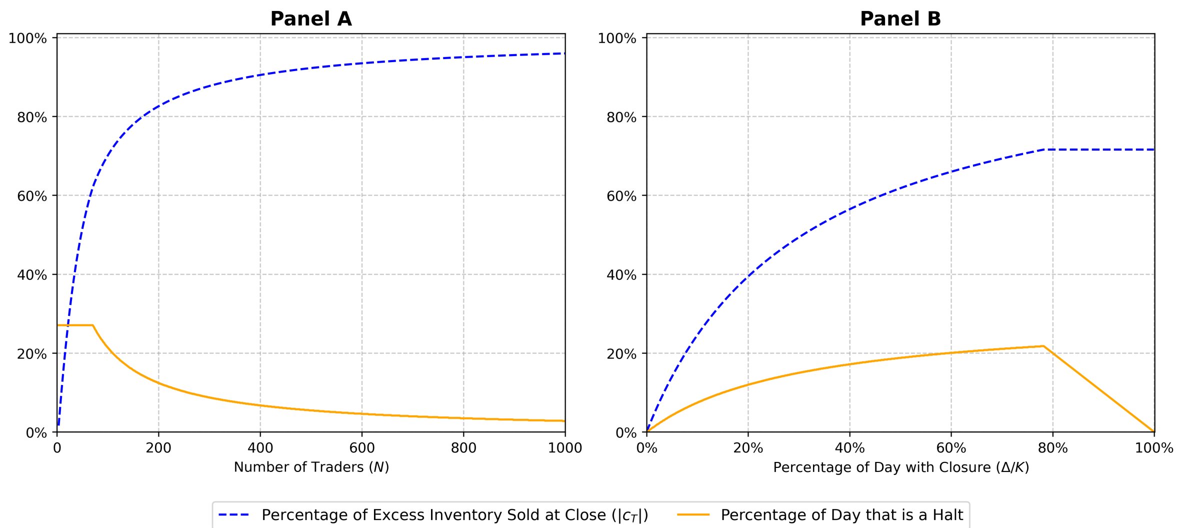


Figure 2. Trading Around the Close

We plot the aggressiveness of traders at the close, $|c_T|$, where closer to 100% is closer to perfect competition, with a blue dotted line, and the percentage of the trading day where no trade endogenously happens leading up to the close with an orange solid line. In Panel A, we plot these two quantities as a function of the market size, N. In Panel B, we plot these two quantities as a function of the percentage of the day where the market is closed, Δ/K. The continuous trade version of the blue-dotted line is equation 11, and the continuous trade version of the solid-orange line is equation 10. We use $r = 1/30$ and $K = 1,000$ for both plots. In Panel A, we set $\Delta/K = 73\%$, and in Panel B, we set $N = 100$.

ness is the same during this time, whether there is a closure or not, which can be seen by the solid and dotted orange lines being indistinguishable. When trade is slower, there is some oscillation in aggressiveness around the level of aggressiveness in the 24/7 trade model (see property 4 of Proposition 1). If liquidity is better next period, agents are less willing to trade now, which lowers aggressiveness and liquidity this period. If liquidity is poor next period, agents are more willing to trade now and incur further price impact. So, the non-stationarity of the trader's problem generates an oscillation that increases in magnitude as the closure approaches. Overall, this oscillation is relatively small in magnitude and can be seen by the dashed blue line alternating below and above the solid blue line.

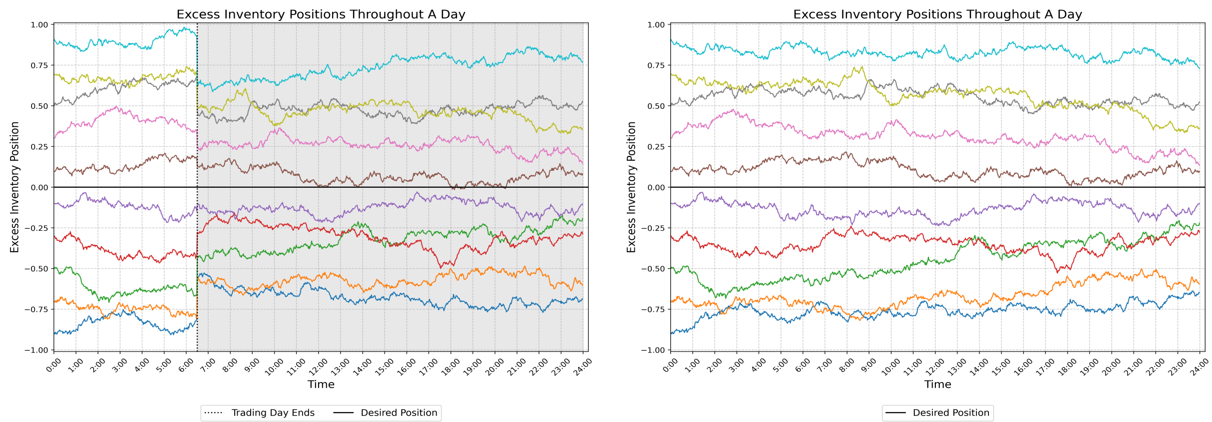
In Figure 2, we study trade aggressiveness at the close and the length of the endogenous no-trade period. We study these two quantities as functions of the number of traders and the length of the overnight closure. The lines in the plot are the discrete-trade versions of equations 10 and 11. Panel A studies how these endogenous quantities change as the market grows in size. First, we measure trade aggressiveness by the fraction of excess inventory that is sold at the close, $|c_T|$, which is plotted as a dotted blue line. The closer this value is to 100%, the closer the model is to perfect competition, and the more efficiently the asset is traded at the close. As the market becomes larger, price impact decreases as demand is dispersed across more traders. Very quickly, the majority of the excess inventory is reallocated in any given period, including the close.

The orange line in Panel A is the length of the no-trade period prior to the closure. For the parameters considered, and when there are fewer than roughly 75 traders, there is no trade apart from at the closing auction. Then, as the number of traders increases, the fraction of the day with endogenously no trade decreases towards zero. As the market grows, price impact decreases, making it less costly to trade in any period before close and minimizing the relative benefits of coordinated liquidity at the close. For sufficiently many traders, the length of the no-trade period is zero by Proposition 1, although this number is not reached in Panel A.

In Panel B, we show that as the length of closure increases, trade aggressiveness and the efficiency of trade at the close increase. As the length of closure increases, so does the willingness of traders to incur price impact at the close. Eventually, the closure is so long that there is only trade at the close, and the line flattens. By similar logic, the length of the no-trade period increases as the efficiency of the closing session improves, as there is a greater incentive to postpone trade. Eventually, there is only trade at the close, which is mechanically moved towards the open for Δ large enough, when the orange line has a slope of -1.

D. A Simulation of the Models

To examine the inventory paths that different market structures induce for traders, we simulate a trading day for a market with ten traders. We run a single simulation for two scenarios: first, when trade occurs for the first 6.5 hours of the day and is followed by a 17.5 hour closure, and second, when trade is 24/7. Each trader receives the same shocks to their inventory position in the two scenarios. The only difference between the two scenarios is the endogenous change in their strategies when there is a daily closure. We set the initial excess



(a) Simulation with $\Delta = 17.5$ hours of night

(b) Simulation with trading 24/7 ($\Delta = 0$)

Figure 3. Simulation With and Without Closure

These figures plot excess inventory paths under the same simulated shocks over a single day for ten traders, $N = 10$, but the left plot has a closure of 17.5 hours, and the right plot allows trade 24/7. The desired excess inventory position (the solid black line) is zero, and the shocks to traders' private values are the same across plots and occur every period right after trade. The parameters used are $\sigma = 1$, $r = 10\%$, $K = 1,000$, and $\gamma = .4$.

inventory positions to be equally spaced between $-.9$ to $.9$ for the $N = 10$ traders.

The results of these simulations are plotted in Figure 3. Let us start with Figure 3(a). While there is noise in the traders' inventory positions during the trading day due to shocks to their desired position, at the close, there is a large drop in the amount of excess inventory held across traders. This drop results from the coordinated trade and liquidity a closure induces.

Using the same shocks, we plot how the trader's excess inventory position would have endogenously evolved in a model with 24/7 trade in Figure 3(b). Without market closure, traders strategically break up their orders over time, spreading out liquidity and trading slowly toward their desired inventory positions. Without the coordination of liquidity a closure provides, traders never substantially close the gap between their current and desired inventories. They do appear to be in better positions by the end of the day, however. From this simulation alone, it is unclear which scenario the traders would prefer ex ante. In Section IV, we will formally study trader welfare as a function of the market structure.

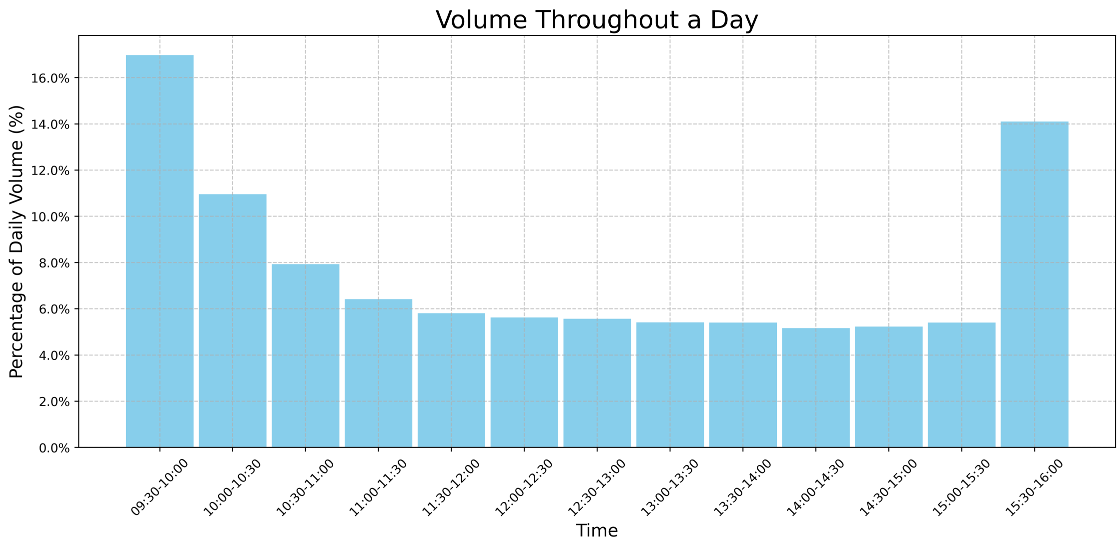


Figure 4. Volume Throughout the Day

This figure is the percentage of the expected daily trading volume in each 30-minute bin when trading occurs for 6.5 hours a day. We simulate 1,000 trading days and plot the average fraction of daily volume in each bin. This example uses $N = 500$, $r = 10\%$, $\sigma = 1$, $K = 1,000$, and $\Delta/K = \frac{17.5}{24}$.

E. Volume

Intraday volume patterns can be used to summarize the intuition of the model. A robust empirical pattern is the U-shaped (smirk) pattern of trading volume throughout the day (e.g., Chan et al. (1996), Jain and Joh (1988)).

Due to the inability to trade overnight, the absolute gap between any trader's current and desired inventory position grows overnight in expectation. Therefore, although trade is not very aggressive in the morning in the sense that traders exchange a small percentage of the gap (small $|c_k|$), due to the large average gap, they still trade a large quantity of the asset. During the middle of the day, traders are neither particularly aggressive nor have a large excess inventory position. Finally, at the close, traders become very aggressive and close the gap significantly, resulting in a large increase in trading volume (see, e.g., Bogousslavsky and Muravyev (2023)).

Figure 4 demonstrates the above reasoning. Figure 4 plots the expected fraction of the total daily volume in each 30-minute trading bucket by computing the average volume in simulations of the model. To match the NYSE, we assume the trading day is 6.5 hours. If trade volume were uniformly distributed throughout the day, you would expect about 7.7% of the daily volume in each bin. Yet, we see significantly more near the open and close. About 17% of the daily volume is clustered in the first 30 minutes, and about 14% is clustered in the last 30 minutes.

IV. Welfare

We now formally study whether traders are better off ex ante in a market structure with a daily closure of some length or in a market structure that allows for 24/7 trade. We do this by studying the aggregate ex-ante welfare of traders. Specifically, we define welfare as the sum of traders' ex-ante expected value functions. As each trader's value function aggregates their expected profits net of inventory costs, the higher its value, the more efficient the market is. In this section, for simplicity, we assume that the initial inventory position for each trader is zero, $z_0^i = 0$, which implies that $\bar{Z} = 0$. We assume each initial private value is i.i.d. $N(0, \sigma^2)$ distributed. We will also focus on the continuous trade version of the model for simplicity. The discrete trade version of the model has qualitatively similar welfare results.

As a first benchmark, we define the first-best (efficient) welfare as that which continuously allocates each trader their inventory position in the competitive benchmark. This benchmark is what a benevolent social planner would achieve if both frictions in the model were eliminated by making trade perfectly competitive and allowing trade to occur continuously and 24/7. Efficient welfare is

$$W^e := \sum_{i=1}^{N} \mathbb{E}\left[V^e(z^i = 0, w^i, \bar{W})\right] = \frac{\sigma^2(N-1)(r+\lambda)}{2\gamma}. \tag{14}$$

Next, we quantify welfare under the market structure with 24/7 trade. The 24/7 welfare is

$$W^{24/7} := \sum_{i=1}^{N} \mathbb{E}\big[V(z^i = 0, w^i, \bar{W})\big] = N\alpha_0 + \sigma^2\bigg(N\alpha_5 + \alpha_6 + \alpha_9\bigg), \tag{15}$$

where the α_i's determine the equilibrium value function, given in Internet Appendix IA.4 when Δ is set to 0. Finally, we quantify the welfare achieved from an equilibrium market structure with a market closure for a fraction Δ of the day. Welfare under a market closure of length Δ is

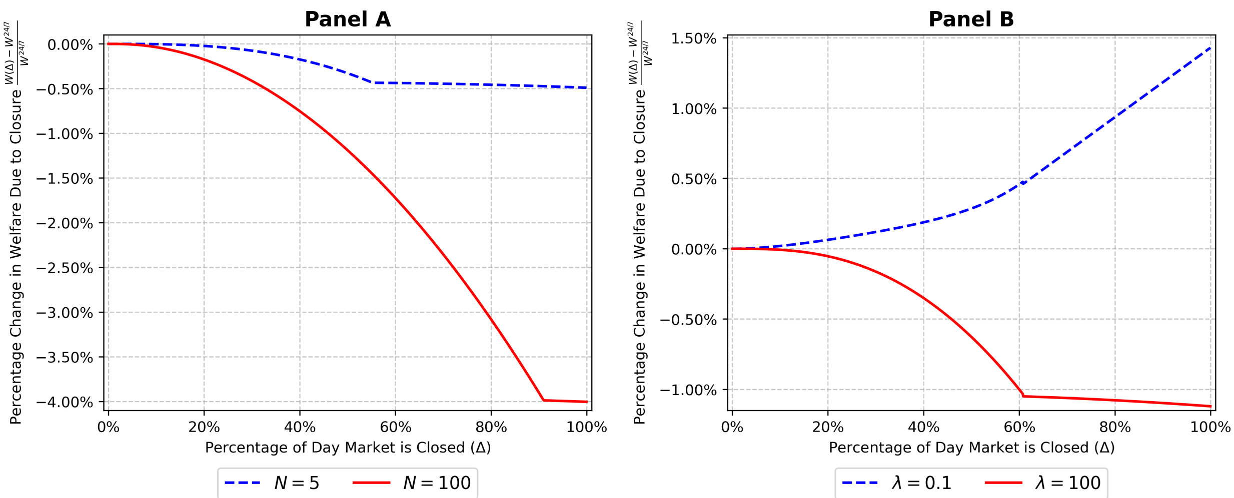


Figure 5. Welfare Comparative Statics

Above is the percent change between welfare under a market closure and welfare under 24/7 trade as we vary the length of the closure in the equilibrium of the continuous trade model. Panel A plots this relationship for two different numbers of traders. Panel B plots this relationship for two different rates of shocks. Both plots use $r = 10\%$. In Panel A, $\lambda = 10$. In Panel B, $N = 10$.

$$W(\Delta) := \sum_{i=1}^{N} \mathbb{E}\left[\frac{1}{1-\Delta}\int_0^{1-\Delta} V_t(z^i = 0, w^i, \bar{W})dt\right]$$
$$= \frac{1}{1-\Delta}\int_0^{1-\Delta} N\alpha_0(t) + \sigma^2\Big(N\alpha_5(t) + \alpha_6(t) + \alpha_9(t)\Big)dt, \quad (16)$$

where the α_i's determine the equilibrium value function, given in Internet Appendix IA.4. Since welfare with a closure is a non-stationary function of time, we compute welfare by averaging across time periods in the trading day. In effect, time is an additional state variable, and, in addition to randomizing across initial values of w^i and $\bar{W}$, we also randomize across the initial time at which the trader begins trading.

A. *Welfare Comparative Statics*

In Figure 5, we plot the percentage change in welfare from a market structure with 24/7 trade to welfare from a market structure with a closure. We display the percentage change as a function of the closure length. Panel A plots the relationship for two different market sizes, and Panel B plots the relationship for two different private value shock arrival rates.

In Panel A, we show that welfare changes are more negative for the larger market, particularly for long closures. In larger markets, the costs of strategic trade are lower. There is not a substantial price impact at any period throughout the day, and, therefore, closure is relatively more costly. In small markets, the benefit of the coordinated trade in the closing session offsets relatively more of the cost of the closure, since liquidity is otherwise spread thin throughout the day. In fact, there is an interior optimal length of closure near 5% of the day. There is also an interior optimal length of closure in the larger market, although it is very small. We will discuss the interior optima further in Section IVB.

In Panel B, welfare differences are displayed for different rates of shocks to private values. If the shocks are infrequent, closure benefits traders. If the frequency of shocks is higher, the lower the relative welfare with a long closure. This is due to the fact that the average gap generated overnight between current and desired inventory widens as the length of closure increases and as the rate of shocks increases. If there are no shocks overnight, then the probability that your inventory position, which tends to be good at the close, is near the desired position at the following open is high. But if there are many shocks at night, then the position you start at the beginning of the next day will be suboptimal, which will be costly to slowly correct in the subsequent trading days. Again, even the case with $\lambda = 100$ has an interior optimal length of closure, although it is small.

We have assumed that the parameters governing the rate of shocks or holding costs are the same overnight as during the trading day, although there may be reason to believe they differ. In Internet Appendix IA.2, we relax this assumption and show welfare moves intuitively as these parameters change from day to night.

B. Is 24/7 Trading Better?

While there is some length of closure that is better than 24/7 trading in Figure 5, it is not obvious whether that is always the case. Proposition 3 shows that there is always a market design with a daily market closure of some length that is strictly better than having trade occur 24/7.

PROPOSITION 3: *There always exists a closure length, $\Delta \in (0, 1)$, such that the ex-ante welfare of a market design with a market closure is greater than that of a market design of 24/7 trading, where welfare is measured by Equation 16.*

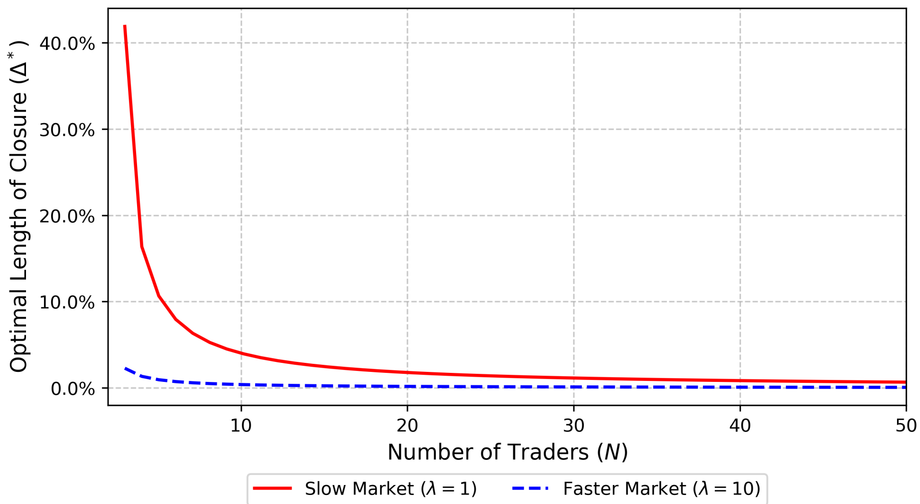


Figure 6. Optimal Length of Closure
We plot the ex-ante welfare maximizing length of closure, Δ^*, that maximizes Equation 16. We assume that σ and γ are constant across day and night and use $r = 10\%$.

The proof is found in Internet Appendix IA.4.4. Within the confines of our model and assumptions, Proposition 3 shows that 24/7 is never optimal for traders, and there is always a benefit of at least a short closure.

How long should the closure be? Proposition 3 gives no guidance on that dimension. While we do not provide closed-form expressions for the optimal length of the closure, Δ^*, we investigate its value numerically in Figure 6. In Figure 6, we plot the optimal length of closure as a function of the size of the market, N. We plot separate lines as a function of the information arrival frequency, λ. The plot shows that in smaller markets, those with fewer traders or slower information arrival, the optimal length of closure can be fairly long at over 40%. However, as the number of traders or the frequency of information arrival increases, the optimal length of closure approaches zero quickly. It is worth noting that it never actually reaches zero but becomes economically equivalent to 24/7 trade in larger markets with a fast rate of information arrival.

Overall, the results of this section and Figure 5 suggest 24/7 trading is near optimal in large markets. Traders in larger markets with frequent shocks to desired positions, such as equities, cryptocurrencies, futures, and foreign exchange markets, are better off in the model with near 24/7 trade. A daily closure is useful in small markets where shocks are infrequent. Asset classes such as corporate bonds or index CDSs fit this description well. In the next section, we calibrate the model to large equity exchanges to study the implications of our model for recent proposals to extend trading hours.

V. Calibration

To apply our model to the data, we calibrate key model parameters for several exchanges. Then, we quantify the welfare gains or losses from changes in trading hours. More specifically, we calibrate the number of traders per exchange, N, and the relative volatility of shocks to private values between the day and night, σ_d/σ_n, allowing this value to differ from 1, as in Internet Appendix IA.2. To estimate these parameters, we match some moments of intraday volume in our model to the data. Given a closure length, the number of traders, and the relative volatility from day to night, the model implies an expected volume in a given time period as a fraction of the total expected volume in a day, as described in Appendix IA.4.1.[7] We match these moments to moments from four different exchanges: NYSE, NYSE Arca, Nasdaq, and CBOE EDGX. We select these four exchanges as the NYSE is the largest registered U.S. equity exchange, and the Nasdaq, CBOE EDGX, and NYSE Arca have announced plans to extend to 24/5, 24/5, and 22/5 trading days, respectively.

We need two linearly independent moments to identify our two parameters. We use the average fraction of daily volume per exchange in the first 3 hours and last 3 hours, which we estimate from TAQ data.[8] The fraction of total volume that is in the first and last 3 hours of trade helps to identify N. If N is smaller, then two moments are closer to summing to 100%. The ratio of instantaneous volatilities, $\frac{\sigma_d}{\sigma_n}$, helps to identify how much volume is in the first 3 hours relative to the last 3 hours. The higher the ratio of instantaneous volatilities, the more volume will concentrate in the last 3 hours of trade, and vice versa. We use the calibrated parameters to study counterfactual daily closure lengths. We fix the total daily private value volatility per exchange to be constant by assuming σ_d solves $\sigma_T^2 = (1-\Delta)\sigma_d^2 + \Delta\sigma_n^2$ so that

[7]The moments we have chosen only identify the relative magnitude and not the level of volatility from day to night. Percentage changes in welfare also depend only on the relative magnitude, not the level. To make the computation of volume more tractable, we use the continuous trade model and assume shocks to private values occur continuously as a Brownian motion. Assuming shocks are Brownian is a limiting case of the jump process for private values as the arrival rate approaches infinity.

[8]The middle section is a linear combination of the other two moments, which provides no new information.

Table I
Calibration

This table compares the welfare of the current market closure to that of 24/7 trading, 23/7 trading, or the optimal length of closure by using the calibrated volatility and number of traders per exchange. $\hat{N}$ denotes the estimated size of the market, and $\hat{\frac{\sigma_d}{\sigma_n}}$ is the relative instantaneous volatilities during the day and night. We assume that total volatility is constant across closure lengths so that σ_d solves $\sigma_T^2 = (1-\Delta)\sigma_d^2 + \Delta\sigma_n^2$. The optimal length of closure, Δ^*, is that which maximizes welfare defined by Equation 16 given the calibrated parameters and subject to the total volatility constraint. We assume $r = 10\%$, $v = 0$, and $z_0^i = 0$ for all calibrations.

Exchange	**Current Length of Night (Δ)**	$\widehat{N}$	$\widehat{\frac{\sigma_d}{\sigma_n}}$	**Optimal Length of Night (Δ^*)**	**% Welfare Change from Δ to 23/7**	**% Welfare Change from Δ to 24/7**	**% Welfare Change from Δ to Δ^***
NYSE	72.9%	208	1.28	0.469%	2.053%	2.057%	2.057%
Nasdaq	72.9%	325	1.32	0.480%	1.997%	2.002%	2.002%
NYSE Arca	72.9%	303	1.23	0.123%	2.128%	2.133%	2.133%
CBOE EDGX	72.9%	191	0.87	0.137%	2.606%	2.612%	2.612%

total volatility is constant as a function of closure lengths.

We estimate the welfare change that would occur if trading were to operate 23/7, as proposed by 24X. This value is also close to the proposed trading hours for NYSE Arca, CBOE EDGX, and Nasdaq. Then, we compare this counterfactual welfare to the estimate of welfare under the current 17.5-hour closure. We also compare the welfare change from the current market structure to 24/7 trade and, finally, from the current to an optimal closure length. The optimal length of closure, Δ^*, is that which maximizes welfare defined by Equation 16 given the calibrated parameters and subject to the total volatility constraint. The results are in Table I.

Table I suggests that, in the model, extending trading hours results in an increase in the welfare (allocative efficiency) of the market. Intuitively, as we have calibrated to large exchanges, the liquidity coordination channel is not as important as the ability to trade for a relatively large fraction of the day since the market is already fairly liquid. Our calibration suggests that the NYSE and other large equity exchanges, such as the London and Tokyo stock exchanges, should consider extending their trading hours. In thinner markets, such as microcap equities, smaller international exchanges, or electronic corporate bond trading, we would expect a calibration to imply that moving to 24/7 trade would decrease trader welfare. Interestingly, the welfare gain comes mostly from extending to 23/7, with only a very small additional gain from going all the way to 24/7 or the optimal length of closure. It is worth noting that the optimal length of closure is an interior length of 2 to 7 minutes a day, which is very short. In Section VII, we further discuss how additional forces may affect these implications.

VI. Heterogeneous Information

In this section, we summarize an extension that allows for heterogeneous fundamental information regarding the dividend. The main results are analogous to those in previous sections, suggesting that our findings regarding the effect of a market closure on liquidity and allocative efficiency are robust to the consideration of informational frictions. The introduction of an information problem is done by adding two components to the model: a stochastic liquidating dividend and private signals regarding its payoff. These components generate a learning problem, discussed below, in addition to the inventory management problem detailed in previous sections.

The liquidating dividend is now assumed to evolve according to a continuous-time random walk. Jumps in the dividend v_t are assumed to coincide with the random jumps in the private value shocks and are $N(0, \sigma_D^2)$ distributed. Each trader receives private signals about these jumps. If a jump in the dividend level occurs at time t, the signal is given by $\hat{S}_t^i = v_t - v_{t-} + \epsilon^i$, where $\epsilon^i \overset{iid}{\sim} N(0, \sigma_\epsilon^2)$. If jumps occurred at dates $t_1 < t_2 < \cdots < t_k < t$, trader i forms a signal $S^i \equiv \sum_{j=1}^{k} \hat{S}_{t_j}^i$ at date t. Assume these normally distributed shocks are all independent of each other and of all other shocks in the model. All other aspects of the model are the same as before.

We focus on daily-periodic, linear, and symmetric strategies and conjecture that equilibrium demand schedules in period k take the following form:

$$D_k^i(z^i, w^i, S^i, p) = a_k + b_k p + c_k z^i + f_k(w^i + AS^i).$$

Based on these demand schedules, in equilibrium, any investor will observe $\bar{W} + A\bar{S}$ directly from the price. Note that there is no time dependence in A in our conjectured demand schedule. Along with the assumption that both the dividend value and private values are random walks, this is an important assumption. If there were time dependence in A, investors' conditional expectations of the dividend would no longer be a simple function of a few state variables, namely w^i, S^i, and $\bar{W} + A\bar{S}$. In particular, time dependence in A would effectively force beliefs to be a state variable of the problem. Any investor i's beliefs would depend on other investors' beliefs, which in turn depend on investor i's beliefs. This loop iterates, leading to an infinite regress of beliefs problem, which the literature on market design in dynamic settings has struggled to resolve.[9]

Given the above demand schedules, each investor solves a learning problem. Traders observe z^i, w^i, S^i and $\bar{W} + A\bar{S}$, from which they infer the level of the dividend. Define the information spanned by these signals to be the information set $\mathcal{I}_i$. Then, conditional beliefs at time $t = kh$ of the value of the dividend are

$$E_t[w_t^i + v_t|\mathcal{I}_i] = w_t^i + B_1 S_t^i + B_2(\bar{W}_t + A\bar{S}_t),$$

for some constants, B_1 and B_2. B_1 and B_2 unsurprisingly depend on A, as the relative weight of the signal from the price on $\bar{W}$ and $\bar{S}$ affects the learning problem. Conversely, A depends on B_1 and B_2, as optimal demand schedules depend on beliefs. This fixed-point problem leads to a straightforward nonlinear equation for A.

We provide the solution to this model in the Appendix B. It is fairly straightforward to show that if the learning problem goes away, in the sense that $B_1 = B_2 = 0$, the equilibrium reduces to that described in Proposition 1. Defining $s^i = \frac{1}{\alpha}(w^i + AS^i)$ for a constant α, with a slight relabelling of the demand function, equilibrium demand is given by

$$D_k^i(p_{kh}^*) = c_k\bigg(z_{kh}^i - \big(\frac{r(N\alpha - 1)}{\gamma(N-1)}(s_{kh}^i - \bar{s}_{kh}) + \bar{Z}\big)\bigg).$$

s_k^i is simply a weighted sum of trader i's private value and their signal. α is an endogenous measure of the amount of adverse selection in the market. When $\alpha = 1$, there is no adverse selection, and traders learn no new information about the asset's payoff from the price. As α decreases, they put more weight on the signal inferred from the market and less on their own information. We will show that the main result of this paper still holds when learning is introduced. As the trading day comes to an end, traders trade aggressively towards their desired allocations. As they do so, price impact decreases, further improving liquidity and the incentives to trade aggressively in the final period.

We plot trading intensity and welfare in Figures 7 and 8. We consider the model of this section alongside two models: one in which σ_ϵ is set to 0, thereby eliminating adverse

[9] See also Du and Zhu (2017b), footnote 6. For recent progress, see Rostek et al. (2025).

selection, and another with adverse selection but without market closure. In Figure 7, we consider trading intensity by plotting $\prod_{j=0}^{k}(1+c_j)$ as a function of k. This quantity measures how much of the gap between a trader's initial inventory and initial desired inventory has closed between the start of the trading day and time t, assuming no shocks have arrived in the interim. For both models with closures, trade is most aggressive in the final period. Perhaps unsurprisingly, trading intensity with adverse selection is slightly lower than without. Traders avoid price impact as purchasing the asset increases others' beliefs about the liquidation value, making them even less willing to sell the asset. It is worth noting that this slower trading is primarily due to heterogeneity, rather than simply uncertainty. In particular, in a model in which signals are public, trading intensity is the same as in a model with no uncertainty about the dividend, due to the risk-neutrality of the traders.

In Figure 8, we see that market closure continues to have consequences for welfare. Welfare is larger with a long closure if the rate of information arrival is sufficiently low. Moreover, if the number of traders is sufficiently small, the results of the left panel suggest a closure of roughly 10% of the day is optimal. Relative to Figure 5, welfare with a market closure is slightly better relative to welfare under 24/7 trade when agents have heterogeneous information. This is not particularly surprising since the coordination a closure provides near the end of the trading day is relatively more important when liquidity is already spread thin due to heterogeneous information. Overall, the primary mechanisms of this paper are present when there is heterogeneous information regarding asset values.

Although not the focus of this paper, it is worth discussing the potential implications the model may have for price efficiency. One can think of price efficiency as the magnitude of a trader's conditional variance of the dividend given their signals and the price, relative to the unconditional variance of the dividend, that is, $\frac{\mathrm{Var}_t(v_t|\mathcal{I}_i)}{\mathrm{Var}(v_t)}$. This value jumps down whenever trading opens, as traders infer information from the price, and increases on average whenever the market closes. Thus, market closure hinders price efficiency simply because prices are not observed overnight, although price efficiency returns to its level with 24/7 trade as soon as the market is reopened and prices are observed. Although worth pointing out, this is not a particularly surprising finding, as the information structure we consider is simple enough to make the model tractable. Extensions in which some traders had higher-quality signals than others might yield interesting results. Implementing extensions with more interesting information structures is not a trivial problem. The infinite regress of

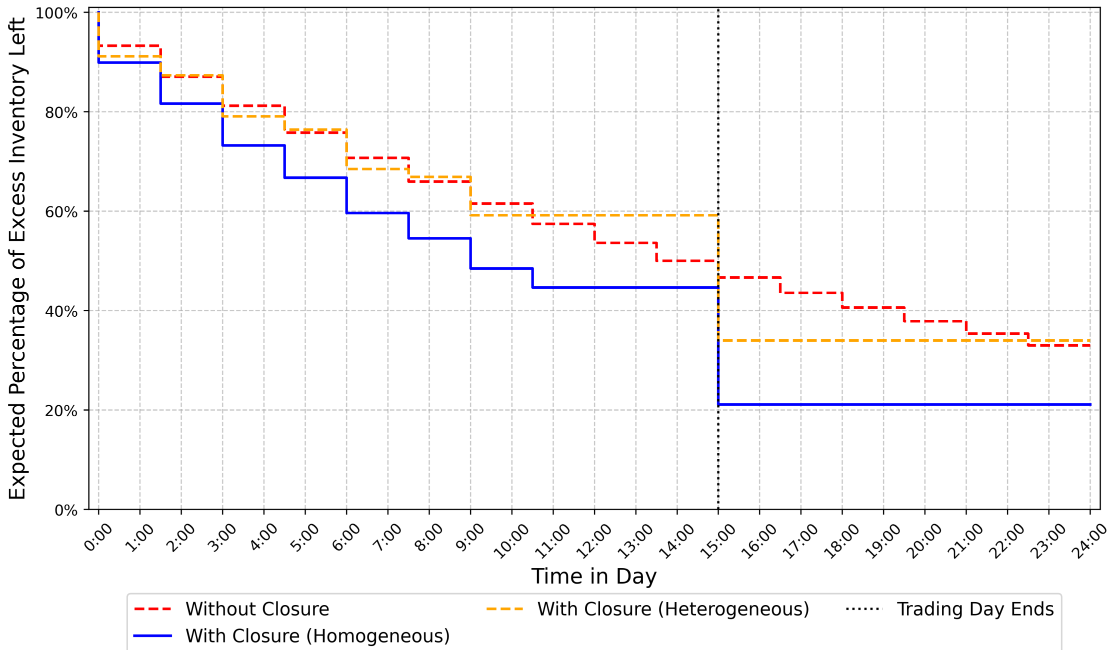


Figure 7. Trading Intensity with Heterogeneous Information

This figure plots trading intensity for various regimes throughout the day. The y-axis is the expected percentage of time 0 excess inventory left at time t in the day. If there is a closure, its length is 31.25% of the day. The parameters are $K = 16$, $N = 100$, and $r = 1/30$. Moreover, $\sigma_D = \sigma = 1$, $\sigma_\epsilon = 0.1$, and $\lambda = 1$. If information is homogeneous, σ_ϵ is set to 0.

beliefs problem mentioned above, which arises even with relatively simple complications of the information structure, makes tractable extensions challenging to formulate. In the absence of these difficulties, the impact of market closure on the dynamic interaction between allocative efficiency, liquidity, and price efficiency with heterogeneously informed investors promises to yield very interesting research, which we leave to future study.

VII. Discussion of Other Policy-Relevant Forces

Our model studies welfare in a model of inventory management and price impact, a first-order consideration in terms of trading costs for large traders. In fact, Frazzini et al. (2018) finds that price impact is the largest trading cost large money managers face, and it leads traders to break up larger orders into child orders that take on average 2.7 days to fully

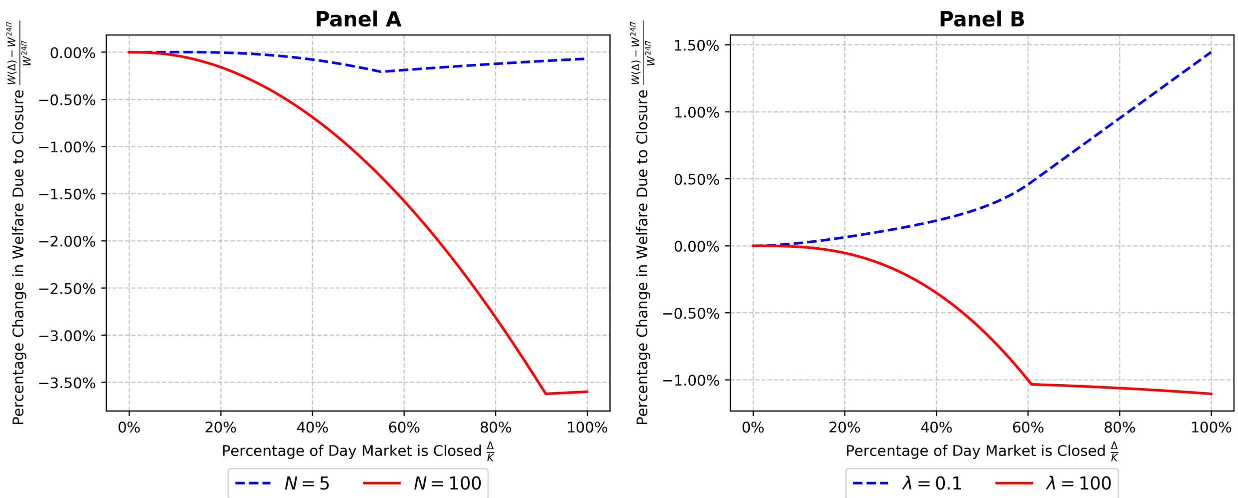


Figure 8. Welfare Comparative Statics with Heterogeneous Information
Above is the percent change between welfare under a market closure and welfare under 24/7 trade as we vary the length of the closure, in the equilibrium of the continuous trade model with heterogeneous information. Panel A plots this relationship for two different numbers of traders. Panel B plots this relationship for two different rates of shocks. Both plots assume $\sigma_D = \sigma = 1$, $\sigma_\epsilon = 0.1$, and $r = 10\%$. In Panel A, $\lambda = 10$. In Panel B, $N = 10$.

execute. In this section, we discuss some additional forces that may be directly relevant for exchanges or policymakers considering modifications to trading hours. Although not explicitly modeled, we argue that the consideration of many of these forces would likely reinforce our main results. Namely, in large equity exchanges, our calibration suggests that extending hours will be beneficial for large traders. More generally, a daily closure coordinates trade, implying the optimal closure is non-zero, and may be large in smaller markets.

A. *Exchanges' Incentives*

The social planner perspective we take allows us to study what traders prefer in counterfactual market structures, and whether a regulator should approve a given change to trading hours. Yet, exchanges ultimately determine their trading hours. To study the incentives of an individual exchange to change its trading hours, we model the exchange as a player who chooses the length of closure to maximize expected volume before any trading occurs.

One major source of revenue for exchanges is volume-related fees. Therefore, the choice of a volume-maximizing closure is an approximation of their decision problem.[10]

Internet Appendix IA.4.2 examines this decision problem. In general, the optimal closures from the perspectives of a trader and an exchange are similar. Volume is maximized with one trading session a day in markets with few traders, but for medium to larger markets, 24/7 trade maximizes volume. Table D.1 studies the decision problem specifically in the calibration of Section V. The calibration suggests that extending trading hours will increase volume, since these exchanges are large. It is worth noting that this result is not entirely obvious. Despite a large percentage increase in trading hours from 6.5 hours a day to 23 or 24 hours a day, the percentage increase in trading volume is much smaller due to the endogenous response of trading strategies.

B. *Market Fragmentation*

While the analysis discussed above suggests that a single exchange's optimal decisions tend to coincide with the optimal market structure for traders, in practice, there are multiple exchanges. This leads to various complications. Exchanges compete with one another,[11] and traders can also split orders across multiple exchanges. Chen and Duffie (2021) model fragmentation in a demand-submission game, and they introduce noise traders on each exchange so that prices across exchanges are imperfectly correlated. They are only able to solve the dynamic model when the number of exchanges is set to its optimal value and trade is efficient (the asset is perfectly reallocated after each trading session). If trade is perfectly efficient, there is no room for the benefits of a daily closure. When trade is not perfectly efficient and there is scope for benefits of a closure, there is the well-known problem of an infinite regress of beliefs, first described in Vayanos (1999), which renders the solution of the model intractable.

Fragmentation may help reduce the risk of holding inventory if exchanges do not perfectly coordinate their trading hours. Indeed, the CME Globex index futures market already operates 23/5, which would allow traders to hedge systematic risk during the current daily

[10] Other revenue sources such as co-location, listing and maintenance fees, and market data fees are all more valuable if volume on that exchange is higher.

[11] The London Stock Exchange Group is considering extending trading hours to near 24/5 to compete with U.S. exchanges, as firms have moved their primary listing to U.S. exchanges, which have already proposed extending trading hours, `https://www.theguardian.com/business/2025/jul/21/london-stock-exchange-24-hour-trading-boost-market`.

market closure. Furthermore, many alternative trading systems (ATSs) operate outside of current market hours, allowing some trading to occur. While this sort of fragmentation is not directly modeled, it could be captured in reduced form through a lower marginal holding cost parameter overnight, γ_n. For instance, hedging in index futures contracts will reduce risk exposures to systematic risks in equities and derivatives markets, although it is important to note that it will not fully eliminate risk. Hedging using index futures, for instance, can be costly and imperfect due to price impact (e.g., Rostek and Yoon (2024)), remaining idiosyncratic risk in equity markets, and remaining unhedgeable risks in derivatives markets (Gârleanu et al. (2009)). Changes in the overnight holding cost are considered in Internet Appendix IA.2. Our calibration in Section V assumes that holding costs are the same overnight as during the day, which is, if anything, a conservative assumption. Increasing the overnight holding cost, which is likely more realistic, would decrease the optimal length of closure, reinforcing our finding that longer trading hours would be welfare-improving for traders on large equity exchanges.

C. Firms

While we focus on the secondary market, many of the firm's actions are influenced by trading hours. Historically, firms prefer to release news and earnings and hold earnings calls outside of trading hours to minimize short-term volatility while investors process information. Many of the exchanges that have proposed extending hours suggest a short daily closure, perhaps in part to accommodate this preference of firms and retain their listings. In our model, shocks to private values or the common dividend value in the heterogeneous information extension can be viewed in part as the result of firm announcements. To account for announcements occurring during a closure, our calibration allows the magnitudes of shocks during the day and night to differ. Moreover, as we vary the length of closure in our calibration, we fix the total daily volatility of shocks constant, so that we effectively consider counterfactuals in which daily firm decisions are held constant. We further study the implications of heterogeneity in volatility between the day and night in Internet Appendix IA.2.

D. Closing Prices

The closing price of a security has become an important component of the financial system. Closing prices are used to calculate the Net Asset Value (NAV) for mutual and open-end funds, for the settlement of many derivative contracts, and for calculating margin and collateral requirements. Moving to 24/7 trading would require redefining this reference point. Our model predicts liquidity is spread more thinly throughout the day when trade is 24/7. Any higher price impact at a new reference point could result in larger price swings, reducing price efficiency. This rationale supports retaining a non-zero daily closure to concentrate liquidity and improve price robustness at the close. We discuss the information structure and its relation to price efficiency in the current model in Section VI.

E. International Access

A common rationale for extending hours is to accommodate demand by international investors. Section VIIA finds that, for liquid exchanges, extending trading hours increases daily volume. However, that analysis does not account for the potential endogenous entry of additional traders if trade were offered during their local business hours. Indeed, recent studies suggest that, at least for some investors, the timing of their trades is related to local business hours. For instance, Eaton et al. (2025) finds that 80% of ATS volume between 8:00 PM and 4:00 a.m. EST comes from Asia-Pacific investors. Further, deHaan and Glover (2024) finds that retail investors just on the west side of a time zone trade less than those just on the east side. In our model, as the number of traders increases, the gains from liquidity coordination decrease, shortening the optimal closure. Endogenizing access would likely strengthen the case for extended trading, though it would also create heterogeneity in agent types, leading again to substantial modeling challenges related to heterogeneity and infinite belief regress (e.g., (Vayanos, 1999)), as the distribution of inventory across types would become a state variable.

F. Regulatory

As noted by many exchanges, extending hours would require regulatory and infrastructure changes. The Depository Trust and Clearing Corp. (DTCC) and securities information processor (SIP) are working to support trading beyond current business hours. These changes are necessary so that securities law, such as Regulation National Market System (Reg NMS), which mostly do not apply overnight, can be followed. For example, the Order Protection Rule (Rule 611 of Reg NMS) does not currently apply overnight, as there is no NBBO disseminated by the SIP.[12] A model with infrastructure adjustment or legal compliance costs would presumably increase the minimum expected profit an exchange would require to be willing to extend trading hours. However, extending regulatory protection, rules, and market infrastructure to non-traditional trading hours could benefit execution quality relative to the current paradigm, in which there is very little oversight and structure for executing overnight orders through ATSs. These countervailing forces suggest the net effect of incorporating regulatory factors in a framework such as ours is ambiguous.

G. Retail

deHaan and Glover (2024) finds that increased trading access leads to excess trading and reduced capital gains for retail investors, though the net welfare effect is not obvious after accounting for, for example, the subjective utility retail traders derive from being able to trade. On the other hand, retail traders already trade overnight through brokers such as Robinhood, Charles Schwab, and Interactive Brokers. Extending legal protections to these trades may improve execution quality.

More generally, over 90% of retail marketable orders are internalized by wholesalers off-exchange (Gensler, 2022), suggesting that changes in exchange hours will primarily affect retail traders indirectly via their effect on wholesalers. If the allocative efficiency gains in our model are passed on to retail traders through better pricing or execution, our calibration suggests that extended trading hours would benefit retail traders on large equity exchanges.

VIII. Conclusion

This paper studies the effect of daily market closures on liquidity and allocative efficiency. Market closures coordinate trade at the end of the trading day, and this coordination generates social benefits that can outweigh the costs of the restrictions closure imposes on trade. Although in our model there is a non-zero length of closure that always improves welfare relative to a market structure with 24/7 trade, for large markets with many traders and frequent shocks to private values, this optimal length of closure is very short. Our calibration suggests that a short closure of a couple of hours or less would improve welfare relative to current trading hours in large equity exchanges.

[12]However, Best Execution and Interpositioning (FINRA Rule 5310) and the Manning Rule (FINRA Rule 5320) do currently apply to overnight trades.

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Appendix

A. *Proof of Propositions 1-2 and Corollary 1*

This appendix proceeds as follows. First, we set up the problem, which describes equilibrium. Then, in Appendix A.1, we show the solution is unique after imposing a trembling-hand refinement and characterize the solution, proving Proposition 1. Appendix A.2 proves Corollary 1, and Appendix A.3 specializes to the case in which $\Delta = 0$, so that there is no overnight period, to prove Proposition 2.

Under the assumption of linear demand schedules, and based on the form of the payoffs, the value function will be linear-quadratic:[13]

$$V_k(z^j, w^j, \bar{W}) = a_0^k + a_1^k z^j + a_2^k w^j + a_3^k \bar{W} + a_4^k (z^j)^2 + a_5^k (w^j)^2 + a_6^k (\bar{W})^2 + a_7^k z^j w^j + a_8^k z^j \bar{W} + a_9^k w^j \bar{W}.$$

First, we characterize its solution.

Let us assume that $b_k > 0$ for now. Then we will address what happens when $b_k = 0$ at the end of this section. The Bellman equation for every time $t = kh$, where $t < T$, is

$$\begin{aligned} V_k(z^j, w^j, \bar{W}) = \max_{D^j} \Bigg\{ & -D^j p_t^* + (1 - e^{-rh})(z^j + D^j)(v + w^j) - \frac{(1 - e^{-rh})\gamma_d}{2r}(z^j + D^j)^2 \\ & + e^{-rh} \Big[a_0^{t+1} + a_1^{k+1}(z^j + D^j) + a_2^{k+1} w^j + a_3^{k+1} \bar{W} \\ & a_4^{k+1}(z^j + D^j)^2 + a_5^{k+1}((w^j)^2 + \lambda\sigma^2) + a_6^{k+1}(\bar{W}^2 + \frac{\lambda\sigma^2}{N}) \\ & + a_7^{k+1}(z^j + D^j) w^j + a_8^{k+1}(z^j + D^j)\bar{W} + a_9^{k+1}(w^j \bar{W} + \frac{\lambda\sigma^2}{N}) \Big] \Bigg\}, \end{aligned}$$

and for the last period, by periodicity, it is

$$\begin{aligned} V_T(z^j, w^j, \bar{W}) = \max_{D^j} \big\{ & -D^j p_T^* + (1 - e^{-rh(1+\Delta)})(z^j + D^j)(v + w^j) \\ & - \frac{(1 - e^{-rh(1+\Delta)})\gamma_n}{2r}(z^j + D^j)^2 + e^{-rh(1+\Delta)} \big[a_0^0 + a_1^0(z^j + D^j) + a_2^0 w^j + a_3^0 \bar{W} \\ & a_4^0 (z^j + D^j)^2 + a_5^0((w^j)^2 + \lambda(1+\Delta)\sigma^2) + a_6^0(\bar{W}^2 + \frac{\lambda(1+\Delta)\sigma^2}{N}) \end{aligned}$$

[13] One can apply a contraction mapping theorem to show the uniqueness of the solution to the trader's decision problem given the other trader's demand functions. First, one can restrict the decision space to a compact subset of the set of linear demand functions. Value iteration will map the set of bounded continuous functions into itself, assuming a Feller-type condition regarding the continuity of the conditional expectation of the continuation value and assuming boundedness is defined using a weighted norm of the form $||f|| = \sup |f(t, z, w, \bar{W}) e^{-||(z,w,\bar{W})||_2^2}|$. Then, using Blackwell's conditions along with the Contraction Mapping Theorem, one gets uniqueness on any compact subset of linear demand functions.

$$+a_7^0(z^j + D^j)w^j + a_8^0(z^j + D^j)\bar{W} + a_9^0(w^j\bar{W} + \frac{\lambda(1+\Delta)\sigma^2}{N})\Big]\Big\}.$$

The FOC for optimal demand in the first $T-1$ periods is then

$$\begin{aligned}0 = -p_t^* - \lambda_k D^j + (1-e^{-rh})(v+w^j) - \frac{(1-e^{-rh})\gamma_d}{r}(z^j + D^j) \\ + e^{-rh}[a_1^{k+1} + 2a_4^{k+1}(z^j + D^j) + a_7^{k+1}w^j + a_8^{k+1}\bar{W}],\end{aligned}$$

and in the last trading session of the day

$$\begin{aligned}0 = -p_T^* - \lambda_T D^j + (1-e^{-rh(1+\Delta)})(v+w^j) - \frac{(1-e^{-rh(1+\Delta)})\gamma_n}{r}(z^j + D^j) \\ + e^{-rh(1+\Delta)}[a_1^0 + 2a_4^0(z^j + D^j) + a_7^0 w^j + a_8^0\bar{W}].\end{aligned}$$

where $\lambda_k := \frac{\partial \Phi_t}{\partial d^j} = -\frac{1}{b_k(N-1)}$. Assume

$$D_k^j = a_k + b_k p_t + c_k z^j + f_k w^j.$$

Market clearing implies the equilibrium price is

$$p_t = -\frac{a_k + c_k\bar{Z} + f_k\bar{W}_t}{b_k},$$

and equilibrium demand is

$$D_k^j = c_k(z_t^j - \bar{Z}) + f_k(w_t^j - \bar{W}_t).$$

Substituting these expressions into the FOC,

$$\begin{aligned}&\frac{a_k + c_k\bar{Z} + f_k\bar{W}}{b_k} + \frac{1}{b_k(N-1)}(c_k(z^j - \bar{Z}) + f_k(w^j - \bar{W})) \\ &\quad + (1-e^{-rh})(v+w^j) - \frac{(1-e^{-rh})\gamma_d}{r}((1+c_k)z^j - c_k\bar{Z} + f_k(w^j - \bar{W})) \\ &\quad + e^{-rh}\left[a_1^{k+1} + 2a_4^{k+1}((1+c_k)z^j - c_k\bar{Z} + f_k(w^j - \bar{W})) + a_7^{k+1}w^j + a_8^{k+1}\bar{W}\right] = 0,\end{aligned}$$

and

$$\begin{aligned}&\frac{a_T + c_T\bar{Z} + f_T\bar{W}}{b_T} + \frac{1}{b_T(N-1)}(c_T(z^j - \bar{Z}) + f_T(w^j - \bar{W})) \\ &\quad + (1-e^{-rh(1+\Delta)})(v+w^j) - \frac{(1-e^{-rh(1+\Delta)})\gamma_n}{r}((1+c_T)z^j - c_T\bar{Z} + f_T(w^j - \bar{W})) \\ &\quad + e^{-rh(1+\Delta)}\left[a_1^0 + 2a_4^0((1+c_T)z^j - c_T\bar{Z} + f_T(w^j - \bar{W})) + a_7^0 w^j + a_8^0\bar{W}\right] = 0.\end{aligned}$$

Grouping common terms,

$$\frac{a_k + c_k\bar{Z}}{b_k} - \frac{c_k\bar{Z}}{b_k(N-1)} + (1-e^{-rh})v + \frac{(1-e^{-rh})\gamma_d c_k\bar{Z}}{r} + e^{-rh}a_1^{k+1} - 2e^{-rh}a_4^{k+1}c_k\bar{Z} = 0,$$

$$\frac{c_k}{b_k(N-1)} - \frac{(1-e^{-rh})\gamma_d(1+c_k)}{r} + 2e^{-rh}a_4^{k+1}(1+c_k) = 0,$$

$$\frac{f_k}{b_k(N-1)} + (1-e^{-rh}) - \frac{(1-e^{-rh})\gamma_d f_k}{r} + 2e^{-rh}a_4^{k+1}f_k + e^{-rh}a_7^{k+1} = 0,$$

$$\frac{f_k}{b_k} - \frac{f_k}{b_k(N-1)} + \frac{(1-e^{-rh})\gamma_d f_k}{r} - 2e^{-rh}a_4^{k+1}f_k + e^{-rh}a_8^{k+1} = 0,$$

and similarly at period T. We show in the Internet Appendix IA.5 that $\alpha_7^k + \alpha_8^k = 1$ and hence $f_k = -b_k$ by the 3rd and 4th FOCs. This leads to the following expressions for the parameters describing demand functions:

$$b_k = \frac{r\big(N-2-(N-1)e^{-rh}(1-a_7^{k+1})\big)}{(N-1)(\gamma_d(e^{-rh}-1)+2re^{-rh}a_4^{k+1})},$$

$$c_k = \frac{2+(a_7^{k+1}-1)e^{-rh} - N(1+e^{-rh}(a_7^{k+1}-1))}{(N-1)(1+e^{-rh}(a_7^{k+1}-1))},$$

$$f_k = \frac{r(1+e^{-rh}(a_7^{k+1}-1))c_k}{\gamma_d(e^{-rh}-1)+2re^{-rh}a_4^{k+1}},$$

$$a_k = -\frac{c_k(N-2)\bar{Z}}{N-1} + b_k\left(v(e^{-rh}-1) - e^{-rh}a_1^{k+1} + \frac{c_k\gamma_d(e^{-rh}-1)\bar{Z}}{r} + 2e^{-rh}c_k\bar{Z}a_4^{k+1}\right).$$

The expression for c_k simplifies to

$$c_k = \frac{1}{(N-1)(1+e^{-rh}(a_7^{k+1}-1))} - 1.$$

Thus, given the coefficients describing the value function, the demand functions are known. Let us now characterize the value function. Returning to the Bellman equation, we have

$$\begin{aligned} V_k = &(c_k(z^j-\bar{Z}) + f_k(w^j-\bar{W}))(\frac{a_k}{b_k} + \frac{c_k}{b_k}\bar{Z} + \frac{f_k}{b_k}\bar{W}) \\ &+ (1-e^{-rh})((1+c_k)z^j - c_k\bar{Z} + f_k(w^j-\bar{W}))(v+w^j) \\ &- \frac{(1-e^{-rh})\gamma_d}{2r}(((1+c_k)z^j - c_k\bar{Z} + f_k(w^j-\bar{W})))^2 \end{aligned}$$

$$+ e^{-rh}\Big[a_0^{t+1} + a_1^{k+1}((1+c_k)z^j - c_k\bar{Z} + f_k(w^j - \bar{W})) + a_2^{k+1}w^j + a_3^{k+1}\bar{W}$$

$$a_4^{k+1}((1+c_k)z^j - c_k\bar{Z} + f_k(w^j - \bar{W}))^2 + a_5^{k+1}((w^j)^2 + \lambda\sigma^2) + a_6^{k+1}(\bar{W}^2 + \frac{\lambda\sigma^2}{N})$$

$$+a_7^{k+1}((1+c_k)z^j - c_k\bar{Z} + f_k(w^j - \bar{W}))w^j$$

$$+a_8^{k+1}((1+c_k)z^j - c_k\bar{Z} + f_k(w^j - \bar{W}))\bar{W} + a_9^{k+1}(w^j\bar{W} + \frac{\lambda\sigma^2}{N})\Big]$$

Matching coefficients in the Bellman equation,

$$a_0^k = -\bar{Z}\frac{c_k a_k + c_k^2\bar{Z}}{b_k} - c_k(1-e^{-rh})v\bar{Z} - \frac{(1-e^{-rh})\gamma_d}{2r}c_k^2\bar{Z}^2$$
$$+ e^{-rh}a_0^{t+1} - e^{-rh}a_1^{k+1}c_k\bar{Z} + e^{-rh}a_4^{k+1}c_k^2\bar{Z}^2 + e^{-rh}a_5^{k+1}\lambda\sigma^2 + e^{-rh}a_6^{k+1}\frac{\lambda\sigma^2}{N} + e^{-rh}a_9^{k+1}\frac{\lambda\sigma^2}{N}$$

$$a_1^k = \frac{c_k a_k + c_k^2\bar{Z}}{b_k} + (1-e^{-rh})(1+c_k)v + \frac{(1-e^{-rh})\gamma_d}{r}(1+c_k)c_k\bar{Z}$$
$$+ e^{-rh}(1+c_k)a_1^{k+1} - 2e^{-rh}(1+c_k)c_k\bar{Z}a_4^{k+1}$$

$$a_2^k = \frac{f_k a_k}{b_k} + \frac{f_k c_k}{b_k}\bar{Z} + (1-e^{-rh})(f_k v - c_k\bar{Z}) + \frac{(1-e^{-rh})\gamma_d}{r}c_k f_k\bar{Z} + e^{-rh}f_k a_1^{k+1} + e^{-rh}a_2^{k+1}$$
$$- e^{-rh}2a_4^{k+1}c_k f_k\bar{Z} - e^{-rh}a_7^{k+1}c_k\bar{Z}$$

$$a_3^k = -\frac{f_k a_k}{b_k} - 2\frac{f_k c_k}{b_k}\bar{Z} - (1-e^{-rh})f_k v - \frac{(1-e^{-rh})\gamma_d}{r}c_k f_k\bar{Z} - e^{-rh}f_k a_1^{k+1} + e^{-rh}a_3^{k+1}$$
$$+ e^{-rh}2a_4^{k+1}c_k f_k\bar{Z} - e^{-rh}a_8^{k+1}c_k\bar{Z}$$

$$a_4^k = -\frac{(1-e^{-rh})\gamma_d}{2r}(1+c_k)^2 + e^{-rh}a_4^{k+1}(1+c_k)^2$$

$$a_5^k = (1-e^{-rh})f_k - \frac{(1-e^{-rh})\gamma_d}{2r}f_k^2 + e^{-rh}a_4^{k+1}f_k^2 + e^{-rh}a_5^{k+1} + e^{-rh}a_7^{k+1}f_k$$

$$a_6^k = -\frac{f_k^2}{b_k} - \frac{(1-e^{-rh})\gamma_d}{2r}f_k^2 + e^{-rh}a_4^{k+1}f_k^2 + e^{-rh}a_6^{k+1} - e^{-rh}a_8^{k+1}f_k$$

$$a_7^k = (1-e^{-rh})(1+c_k) - \frac{(1-e^{-rh})\gamma_d}{r}(1+c_k)f_k + 2e^{-rh}a_4^{k+1}(1+c_k)f_k + e^{-rh}a_7^{k+1}(1+c_k)$$

$$a_8^t = \frac{c_k f_k}{b_k} + \frac{(1-e^{-rh})\gamma_d}{r}(1+c_k)f_k - 2e^{-rh}a_4^{k+1}(1+c_k)f_k + e^{-rh}a_8^{k+1}(1+c_k)$$

$$a_9^t = f_k\left(\frac{f_k}{b_k} - (1-e^{-rh}) + \frac{(1-e^{-rh})\gamma_d}{r}f_k - 2e^{-rh}a_4^{k+1}f_k - e^{-rh}a_7^{k+1} + e^{-rh}a_8^{k+1}\right) + e^{-rh}a_9^{k+1}$$

Now let us address what happens when $b_k = 0$. In this case, trader j is allocated 0 demand in equilibrium in period k, and it is straightforward to see that the value function coefficients obey the same set of recursive equations as above, simply with $a_k, c_k, f_k, f_k/b_k$ all set to 0.

The joint solution of these recursions, along with the expressions for a_k, b_k, c_k, f_k, characterizes the equilibrium. We give simplifications of these recursions in Internet Appendix IA.5.

A.1. Uniqueness, Existence and Properties of the Equilibrium Solution

In this section, we begin by providing properties of a_7, c_k, and f_k that must be satisfied for trade to occur at a date k. These properties show that the existence of a non-zero trade equilibrium at date k can be reduced to a simple condition on $a_7^{k+1(\text{mod } T)}$.

Using this condition, we formulate a recursion for a_7 and show it has a unique solution.

There can be other equilibria, but by the unique solution to this recursion, they must occur in periods in which there is also an equilibrium in which trade occurs. We then proceed to show that these equilibria do not satisfy our trembling-hand refinement.

Last, we conclude by providing some additional properties of the solution, which prove properties 1-4 in Propostion 1.

Properties required for trade in linear demand schedules:

This section first shows that the equilibrium c, f and a_7, a_4 must satisfy certain restrictions for trade to occur at a given date, then discusses the existence of the solution.

Suppose we are at period $k < T$. The case $k = T$ is analogous. We occasionally drop k subscripts to ease notation. Suppose all other traders submit demand curves with negative slope b_k at period k.

The SOC for demand optimization is given by

$$\frac{1}{b(N-1)} - \frac{(1-e^{-rh})\gamma_d}{r} + 2e^{-rh}a_4^{k+1} < 0$$

First, note since $f/b = -1$, we must have $f > 0$ in equilibrium. By the third FOC above, this fact combined with the SOC implies

$$(1-e^{-rh}) + e^{-rh}a_7^{k+1} > 0.$$

Then, by the expression for f, both c and $\gamma_d(e^{-rh}-1) + 2re^{-rh}a_4^{k+1}$ must have the same sign. Now, the second FOC implies

$$\frac{c_k}{b_k(N-1)} = \frac{(1-e^{-rh})\gamma_d(1+c_k)}{r} - 2e^{-rh}a_4^{k+1}(1+c_k).$$

If $c \leq -1$, the LHS is positive while the RHS is negative. So $c \geq -1$. This fact implies, as hinted at in discussions above, that $a_7 > 0$ in an equilibrium with trade at k.

Now since c and $\gamma_d(e^{-rh} - 1) + 2re^{-rh}a_4^{k+1}$ have the same sign, we can analyze $\gamma_d(e^{-rh} - 1) + 2re^{-rh}a_4^{k+1}$ to determine the sign of c. Let's consider the case $k = 0$. Other cases are similar.

$$
\begin{aligned}
a_4^0 &= -\frac{(1-e^{-rh})\gamma_d}{2r}(1+c_0)^2 + e^{-rh}a_4^1(1+c_0)^2 \\
&= -\frac{(1-e^{-rh})\gamma_d}{2r}(1+c_0)^2 - \frac{(1-e^{-rh})\gamma_d}{2r}e^{-rh}(1+c_0)^2(1+c_1)^2 + e^{-2rh}a_4^2(1+c_0)^2(1+c_1)^2 \\
&= \cdots \\
&= -\frac{(1-e^{-rh})\gamma_d}{2r}\sum_{t=0}^{k} e^{-trh}\prod_{i=0}^{t}(1+c_i)^2 + e^{-(k+1)rh}a_4^{k+1}\prod_{i=0}^{k+1}(1+c_i)^2
\end{aligned}
$$

for $k \leq T-1$. Iterating to $k = T$ and beyond is similar. In order for positions to be non-explosive functions of past positions, based on the expression for equilibrium demand, we only consider equilibria that imply $\prod_{i=0}^{k}(1+c_i) \to 0$ as $k \to \infty$. Note that this also implies, taking the limit in the expansion above, that $a_4^0 < 0$. One can show $a_4^k < 0$ similarly.

This, in turn, implies $c < 0$ if trade occurs at period k. It's worth noting that $c < 0$ will imply $\prod_{i=0}^{k}(1+c_i) \to 0$, where one imposes periodicity in the limit in the obvious way.

Thus, we've shown that in equilibrium, a_7 must be positive, and c must be between -1 and 0.

Conversely, if solutions with positive a_7 and c between 0 and -1 exist and satisfy the FOCs above, this then allows for a unique solution for a_4, since its recursion is linear. Solutions of a_7, c, a_4 yield solutions for b and f. The corresponding solution for b_k will be negative in periods in which c_k is negative, since we show below that the recursions above imply c_k/b_k is a positive constant. Moreover, we show below that $a_k/b_k = v$, implying a solution for b yields solutions for a. The remaining recursions for the value function are linear and have simple unique solutions. To sum up, solving the model reduces to solving for a_7.

Existence and uniqueness of a particular equilibrium:

Let us adjust the recursions for a_7 above to be more precise regarding in which periods there is no trade. Once the equilibrium value of a_7 is determined, the rest of the parameters determining the equilibrium can be pinned down as described above. In particular, we will frequently make use of the fact that the expression for c_k in periods with trade must be the same function of a_7^{k+1} as before.

Note if $\frac{1}{(N-1)(1-e^{-rh}+e^{-rh}a_7^{k+1})} > 1$, the arguments above imply there is no solution for trade at k in downward sloping demand curves, because the solution for c_k and hence b_k would be positive. This condition reduces to

$$a_7^{k+1} < 1 - \frac{N-2}{e^{-rh}(N-1)},$$

so that we can redefine the recursion for a_7^{k+1} when we don't require trade every period to

$$a_7^k = \begin{cases} 1 - e^{-rh} + e^{-rh}a_7^{k+1} & \text{if } a_7^{k+1} \leq 1 - \frac{N-2}{e^{-rh}(N-1)} \\ \frac{1}{(N-1)^2(1-e^{-rh}+e^{-rh}a_7^{k+1})} & \text{if } a_7^{k+1} > 1 - \frac{N-2}{e^{-rh}(N-1)} \end{cases}$$

$$= \min\{1 - e^{-rh} + e^{-rh}a_7^{k+1}, \frac{1}{(N-1)^2(1-e^{-rh}+e^{-rh}a_7^{k+1})}\}.$$

Define f to be the right-hand side of this expression as a continuous, piecewise-defined function a_7^{k+1}. This recursion corresponds to the model in which, if there is no trade equilibrium at k in downward-sloping demand curves, there is no trade at k. If there is a trade equilibrium, the corresponding value of a_7^k is selected.

Moreover, f is a contraction from $[0, \infty)$ to itself. One can argue this as follows. f is increasing if $a_7^{t+1} \leq 1 - \frac{N-2}{e^{-rh}(N-1)}$ and decreasing otherwise. Moreover, on the first region, its slope is e^{-rh} and on the second region, its slope is decreasing and maximized when $a_7^{t+1} = \max\{1 - \frac{N-2}{e^{-rh}(N-1)}, 0\}$. Its slope at this point is also strictly less than 1. Therefore, it is straightforward to see that f is a contraction.

Then, we can iterate the recursions for a_7 T times to write a_7^t as the solution of a fixed point problem, by periodicity. Note that the recursion at period T must be appropriately adjusted to account for the overnight period. This fixed point function is the composition of functions that are contractions, and hence a_7^t is the fixed point of a contraction mapping. Thus, by the Contraction Mapping Theorem, there's a unique solution to a_7^t and, therefore, the sequence of a_7's.

Hence, there is a unique solution to the problem for which trade at any period is only abandoned if there is no equilibrium in downward-sloping demand curves in that period. Below, we show that the multiple equilibria that arise when traders can submit demand curves equal to 0, even when there are equilibria that involve trade in that period, do not survive the trembling-hand refinement. Therefore, the fixed point problem described in this section characterizes the equilibrium of this paper.

Trembling-hand refinement:

In this section, we will show that a form of the trembling-hand refinement rules out equilibria in which players choose to submit uniformly 0 demand curves in periods in which there is also an equilibrium in which trade occurs. In conjunction with the uniqueness result of the previous section, this implies that under the refinement, there must be a unique equilibrium.

Consider the optimization problem of investor j at a period $t < Th$. The case $t = Th$ is similar. Suppose there are two equilibria at time t, one with zero trade (and uniformly 0 submitted demand curves), and the other with non-zero trade. Suppose that with probability q the $N-1$ other traders play the non-zero trade equilibrium in the current and future periods. With probability $1-q$, the other $N-1$ traders play the equilibrium with 0 at time t in current and future periods. The trembling-hand refinement we consider will require that as q goes to 0, the optimal demand submission of trader j at time t also converges to a demand submission uniformly equal to 0.

We consider a trembling-hand refinement in which players tremble simultaneously. If players tremble independently, there is a non-zero probability that only one player submits a non-zero demand curve. In two-player demand submission games, the non-existence of linear equilibria is well-known (e.g., Du and Zhu (2017a)). To avoid these issues, we assume simultaneous trembling.

The optimization problem for every time $t = kh$, where $t < T$, is

$$\max_{D^j} \left\{ E\left[-D^j p + (1-e^{-rh})(z^j + D^j)(v + w^j) - \frac{(1-e^{-rh})\gamma_d}{2r}(z^j + D^j)^2 \right.\right.$$
$$\left.\left. + e^{-rh}\left[\tilde{V}_{k+1}(z^j + D^j, w^j_{(k+1)h}, \bar{W}_{(k+1)h}) \right] \right] \right\},$$

where the expectation is over which equilibrium the other traders choose, and $\tilde{V}_{k+1}$ denotes that the continuation value may differ depending on the equilibrium chosen by other players. Note, demand curves are only contingent on the price, not, for instance, which equilibrium other players might tremble to.

Allocations are constrained such that the market clears. Therefore, we can formulate the optimization problem in terms of choosing the price based on the residual demand curve to simplify the exposition. In particular, trader j's allocation is constrained to be the residual demand of the other $N-1$ traders, $D^j = -\sum_{i\neq j} D^i(p)$. Therefore, how much they demand is equivalent to the price they choose. Using this, the optimization problem becomes

$$
\begin{aligned}
V_k(z^j, w^j, \bar{W}) = \max_p \Bigg\{ E \Bigg[& p \sum_{i\neq j} D^i(p) + (1-e^{-rh})(z^j - \sum_{i\neq j} D^i(p))(v + w^j) \\
& - \frac{(1-e^{-rh})\gamma_d}{2r}(z^j - \sum_{i\neq j} D^i(p))^2 + e^{-rh} \Bigg[\tilde{a}_0^{t+1} + \tilde{a}_1^{k+1}(z^j - \sum_{i\neq j} D^i(p)) + \tilde{a}_2^{k+1} w^j + \tilde{a}_3^{k+1}\bar{W} \\
& \tilde{a}_4^{k+1}(z^j - \sum_{i\neq j} D^i(p))^2 + \tilde{a}_5^{k+1}((w^j)^2 + \lambda\sigma^2) + \tilde{a}_6^{k+1}(\bar{W}^2 + \frac{\lambda\sigma^2}{N}) \\
& + \tilde{a}_7^{k+1}(z^j - \sum_{i\neq j} D^i(p))w^j + \tilde{a}_8^{k+1}(z^j - \sum_{i\neq j} D^i(p))\bar{W} + \tilde{a}_9^{k+1}(w^j\bar{W} + \frac{\lambda\sigma^2}{N}) \Bigg] \Bigg] \Bigg\},
\end{aligned}
$$

where again, tildes denote uncertainty about the continuation value. In the equilibrium in which all other traders submit 0, $\sum_{i\neq j} D^i(p) = 0$, so that the problem simplifies to

$$
\begin{aligned}
\max_p \Bigg\{ q \Bigg(& p \sum_{i\neq j} D^i(p) + (1-e^{-rh})(z^j - \sum_{i\neq j} D^i(p))(v + w^j) \\
& - \frac{(1-e^{-rh})\gamma_d}{2r}(z^j - \sum_{i\neq j} D^i(p))^2 + e^{-rh} \Bigg[a_0^{t+1} + a_1^{k+1}(z^j - \sum_{i\neq j} D^i(p)) + a_2^{k+1} w^j + a_3^{k+1}\bar{W} \\
& a_4^{k+1}(z^j - \sum_{i\neq j} D^i(p))^2 + a_5^{k+1}((w^j)^2 + \lambda\sigma^2) + a_6^{k+1}(\bar{W}^2 + \frac{\lambda\sigma^2}{N}) \\
& + a_7^{k+1}(z^j - \sum_{i\neq j} D^i(p))w^j + a_8^{k+1}(z^j - \sum_{i\neq j} D^i(p))\bar{W} + a_9^{k+1}(w^j\bar{W} + \frac{\lambda\sigma^2}{N}) \Bigg] \Bigg) \Bigg\},
\end{aligned}
$$

where the continuation value coefficients correspond to the equilibrium with non-zero trade today. This is then simply the optimization problem in the equilibrium with trade scaled by q. Thus, trader j's demand submission will simply be the same as that in the equilibrium with non-zero trade, irrespective of q, as long as $q > 0$. Intuitively, since their allocation is independent of the demand schedule played in the zero-trade equilibrium, the trader will behave as if the tremble equilibrium is all that matters, as that is the only instance in which their submitted demand schedule matters. One can repeat the arguments above with very slight modifications to show that the equilibrium with trade, when it exists, does in fact satisfy the trembling hand refinement.

Properties of the equilibrium:

It will facilitate the exposition to define quantities a_7^k for $k = T+1, T+2, \ldots, T+\Delta$, where $a_7^k = 1 - e^{-rh} + e^{-rh} a_7^{(k+1)(\text{mod } T+\Delta+1)}$, for $k = T+1, T+2, \ldots, T+\Delta$. Then $a_7^T = \frac{1}{(N-1)^2(1-e^{-rh}+e^{-rh}a_7^{T+1})}$ if there is trade at T and $1 - e^{-rh} + e^{-rh} a_7^{T+1}$ otherwise.

The ordering of the properties here is different than the listing of the properties in Proposition 1. We begin by describing these differences.

Properties (1) and (2) in this section provide some basic properties regarding the solution. We refer to these properties as "oscillation" properties throughout. They prove property 4 in Proposition 1.

Properties (3), (4), (5) show properties (1) and (2) in Proposition 1, by showing that there must be a contiguous sequence of periods with trade, followed by a sequence of periods without trade, and followed by trade at T.

Properties (6) and (7) prove some simplifications of the solution which lead to the expressions in property 3 of Proposition 3.

(1): The first property is that if one a_7^k is larger than the long-run solution (i.e., the solution in which the market is always open), the "next" one, a_7^{k-1}, must be smaller. To see this, define

$$f(x) = \frac{1}{(N-1)^2(1-e^{-rh}+e^{-rh}x)}.$$

The long-run solution solves the quadratic equation given by $f(x_0) = x_0$. Since for $x > 0$, f is decreasing in x, if $x > x_0$, $y \equiv f(x) < f(x_0) = x_0$. So the next iteration y is less than x_0. The opposite happens if $x < x_0$. So solutions oscillate around the long-run solution when the market is open.

(2): Second, we show that the size of the oscillations decreases as one gets further away from the end of trade. To do this, note if $a_7^k = x$, where $k \neq 0, 1$,

$$a_7^{k-2} = f(f(x)).$$

Note the long run solution x_0 solves the quadratic equation $x_0 = f(f(x_0))$. After simplifying, we can write this equation as

$$0 = 1 - (1 - e^{-rh})(N-1)^2 x_0 - e^{-rh}(N-1)^2 x_0^2.$$

Note that the long-run solution x_0 we care about is the positive root. It is straightforward to show, as with our solution for a_7 above, that one root is positive and one is negative, and the quadratic function defined by the right-hand side above is decreasing in the positive reals. In particular, if $0 < x < x_0$,

$$1 - (1 - e^{-rh})(N-1)^2 x - e^{-rh}(N-1)^2 x^2 > 0,$$

which by reversing the same operations that led us from $f(f(x_0)) = x_0$ to the quadratic equation, implies $f(f(x)) > x$, so that $a_7^{k-2} > a_7^k$. Similarly, if $x > x_0$, then $a_7^{k-2} < a_7^k$. So the oscillations decrease in magnitude as one moves further from the end of trade.

We illustrated these first two properties for a_7. The correspondence between a_7 and c and b implies analogous results for c.

(3): Let us spend some time characterizing when trade will occur. Note, by the above, for trade to occur at $k \in \{0, \ldots, T\}$, we need

$$a_7^{k+1} > \left(1 - \frac{N-2}{e^{-rh}(N-1)}\right).$$

Denote the right-hand side of this inequality by a_7^l. If $a_7^l < 0$, trade will occur in every trading period, so we can assume $a_7^l \geq 0$. Now, note if trade occurs at $k+1$,

$$a_7^{k+1} = \frac{1}{(N-1)^2(1 - e^{-rh} + e^{-rh} a_7^{k+2})}.$$

Then, re-writing the inequality above by substituting this expression for a_7^{k+1}, we must have

$$a_7^{k+2} > \frac{\frac{1}{N-1} - ((N-1)(1-e^{rh}) + e^{rh})(1 - e^{-rh})}{((N-1)(1-e^{rh}) + e^{rh})e^{-rh}}$$

for trade to occur at k if it occurred at $k+1$. This inequality uses the assumption that $a_7^l \geq 0$. Then, call the right-hand side of this inequality a_7^h.

Now, note if $a_7^{k+1} \in [a_7^l, a_7^h]$, trade will occur at all periods in the day, using the oscillation properties shown above. To see this, note if trade occurred at k and $\bar{a}_7 < a_7^{k+1} \leq a_7^h$, then $\bar{a}_7 > a_7^k > a_7^l$, so trade will occur at $k-1$, and so $a_7^{k-\ell} > a_7^l$ for all earlier periods in the day. Similarly, if trade occurs at k and $\bar{a}_7 > a_7^{k+1} > a_7^l$, then by the oscillation properties, and $a_7^{k+1-\ell} > a_7^l$ for all earlier periods in the day.

Moreover, if there is trade in two consecutive periods, there must be trade in all prior periods. This is because if there is trade in two consecutive periods, it must be the case that $a_7^{k+1} \in [a_7^l, a_7^h]$ if k is the first of the two periods. If $a_7^{k+1} < a_7^l$, there can't have been trade at k, and if $a_7^{k+1} > a_7^h$, by the oscillation property, $a_7^{k+2} < a_7^k < a_7^l$, and so there can't have been trade at $k+1$.

(4): Now let us show that there cannot be a period without trade followed by a period with trade followed by another period without trade. In other words, if there is trade in a period which is followed by a period without trade, all prior periods must have non-zero trade. Note if there isn't trade in period $k \geq 2$, we must have $a_7^{k+1} < \left(1 - \frac{N-2}{e^{-rh}(N-1)}\right)$ and so $a_7^k = 1 - e^{-rh} + e^{-rh}a_7^{k+1} < \frac{1}{N-1}$. If there is trade in period $k-1$, then

$$a_7^{k-1} = \frac{1}{(N-1)^2(1 - e^{-rh} + e^{-rh}a_7^k)}.$$

And, there will then be trade in $k-2$ since

$$a_7^{k-1} = \frac{1}{(N-1)^2(1 - e^{-rh} + e^{-rh}a_7^k)} > \frac{1}{(N-1)^2(1 - e^{-rh} + e^{-rh}\frac{1}{N-1})} > \left(1 - \frac{N-2}{e^{-rh}(N-1)}\right),$$

where the last inequality is straightforward to verify, as it is equivalent to

$$\frac{(N-2)(e^{rh}-1)^2}{(N-1)e^{rh} - N + 2} > 0.$$

Hence, if there is no trade for a period and there is trade in the preceding period, there is also trade in the preceding two periods, which, by the results in property (4), implies there is trade in all preceding periods.

(5): Let us now show that there must be trade at period T. First, there must be trade in at least one period. Otherwise, $a_7^k = 1 - e^{-rh} + e^{-rh}a_7^{(k+1)(\text{mod } T+\Delta+1)}$ for all k, which implies a_7^k is strictly monotonic in k, unless it always equals 1. Strict monotonicity cannot occur since the solution must be periodic. The solution cannot always equal 1 because $1 > a_7^l$, a contradiction to the assumption of no trade.

Assume there is no trade at $T, T-1, \ldots, T-\ell+1$, but there is trade at period $T-\ell$. Then, by the above arguments, there must be trade at $0, 1, \ldots, T-\ell-1$ as well. As a result, $a_7^0 > a_7^l$. Yet, then $a_7^{T+\Delta} = (1-e^{-rh}) + e^{-rh}a_7^0 > a_7^\ell$, $a_7^{T+\Delta-1} = (1-e^{-rh}) + e^{-rh}a_7^{T+\Delta} > a_7^\ell$, etc., until $a_7^{T+1} > a_7^l$. But this implies there must be trade at T, a contradiction.

(6): Next we show $c_k/f_k = -\gamma/r$. First, recall

$$a_7^k = (1-e^{-rh})(1+c_k) - \frac{(1-e^{-rh})\gamma_d}{r}(1+c_k)f_k + 2e^{-rh}a_4^{k+1}(1+c_k)f_k + e^{-rh}a_7^{k+1}(1+c_k).$$

Plugging in the expression for f_k derived above, this implies

$$a_7^k = (1-e^{-rh})(1+c_k)^2 + e^{-rh}a_7^{k+1}(1+c_k)^2.$$

Thus, defining $\kappa_k = \frac{2r}{\gamma_d}a_4^k + a_7^k$, we have $\kappa_k = e^{-rh}\kappa_{k+1}(1+c_k)^2$ for $t < T$, and similarly when $t = T$. Note this recursion also holds in periods in which there is no trade. This periodic recursion has unique solution $\kappa_k = 0$. Then, the expression for f_k implies $f_k = -\frac{r}{\gamma}c_k$.

(7): The last property is that $a_k/b_k = -v$. Recall the first FOC for optimal demand is

$$\frac{a_k + c_k\bar{Z}}{b_k} - \frac{c_k\bar{Z}}{b_k(N-1)} + (1-e^{-rh})v + \frac{(1-e^{-rh})\gamma_d c_k\bar{Z}}{r} + e^{-rh}a_1^{k+1} - 2e^{-rh}a_4^{k+1}c_k\bar{Z} = 0,$$

By the third FOC above, this can be rewritten as

$$0 = \frac{a_k}{b_k} + \frac{c_k}{b_k}\bar{Z} + \frac{c_k\bar{Z}}{f_k}e^{-hr}a_7^{k+1} + (1-e^{-rh})(v + \frac{c_k\bar{Z}}{f_k}) + e^{-rh}a_1^{k+1}.$$

Then, the recursions for a_1, a_7 imply

$$-\frac{r}{\gamma\bar{Z}}a_1^k + a_7^k = \frac{r}{\gamma\bar{Z}}\left(\frac{a_k}{b_k} + \frac{c_k\bar{Z}}{b_k}\right).$$

Combined, these last two expressions imply

$$-\frac{r}{\gamma\bar{Z}}a_1^k + a_7^k = -\frac{r}{\gamma\bar{Z}}(1-e^{-rh})\left(v - \frac{\gamma}{r}\bar{Z}\right) + e^{-rh}(a_7^{k+1} - \frac{r}{\gamma\bar{Z}}a_1^{k+1}).$$

It's straightforward to show this relation also holds when there is no trade, implying $-\frac{r}{\gamma\bar{Z}}a_1^k + a_7^k = -\frac{r}{\gamma\bar{Z}}(v - \frac{\gamma}{r}\bar{Z})$. Plugging this back into the simplified FOC above, we arrive at $\frac{a_k}{b_k} = -v$.

A.2. Corollary 1: Trade every period

Let us show $(N-1)(1-e^{-rh}) > 1$ is a sufficient condition for an equilibrium with trade every period to exist. Assume we are considering whether there is trade in period $k < T$. Period T is analogous. Since we must have

$$c_k = \frac{1}{(N-1)(1+e^{-rh}(a_7^{k+1}-1))} - 1,$$

and the solution a_7 of the fixed point problem is nonnegative, $c_k \geq -1$. And, $c_k < 0$ if $(N-1)(1-e^{-rh}) > 1$. Hence, $-1 \leq c_k < 0$ every period in which the market is open. Then by the expressions for a_4 given above, the solution for a_4 will be negative. So, f will be positive, and b will be negative, given by the solutions to the first-order conditions above. Therefore, there is an equilibrium with trade every period.

Explicit Solution for $(a_7^k)_{k=0}^T$:

In fact, in the case in which there is trade every period, we can express the solution for $(a_7^k)_{k=0}^T$ in terms of the solution to a quadratic equation.

$$a_7^k = \frac{1}{(N-1)^2(1+e^{-rh}(a_7^{k+1}-1))},$$

for $k = 0, \dots, T-1$. Then, at time T,

$$a_7^T = \frac{1}{(N-1)^2(1+e^{-r(1+\Delta)h}(a_7^0-1))}.$$

Set $a_7^0 = d$ for some constant d which solves a quadratic equation. Write $\delta = e^{-rh}$. The constant term in the quadratic equation is

$$\begin{aligned}
&-2\left((-1+\delta)(N-1)^2 - \sqrt{(N-1)^2\left(4\delta+(-1+\delta)^2(N-1)^2\right)}\right)^{T+1} \\
&+2\left((-1+\delta)(N-1)^2 + \sqrt{(N-1)^2\left(4\delta+(-1+\delta)^2(N-1)^2\right)}\right)^{T+1} \\
&+\delta^{T+1+\Delta}\Bigg[\left((-1+\delta)\delta^{-(T+1)} - (-1+\delta)\delta^{-(T+1+\Delta)}\right)(N-1)^2 \\
&\quad\times\Bigg(\left((-1+\delta)(N-1)^2 - \sqrt{(N-1)^2\left(4\delta+(-1+\delta)^2(N-1)^2\right)}\right)^{T+1} \\
&\quad-\left((-1+\delta)(N-1)^2 + \sqrt{(N-1)^2\left(4\delta+(-1+\delta)^2(N-1)^2\right)}\right)^{T+1}\Bigg) \\
&\quad+\left(\delta^{-(T+1)} - \delta^{-(T+1+\Delta)}\right)\sqrt{(N-1)^2\left(4\delta+(-1+\delta)^2(N-1)^2\right)} \\
&\quad\times\Bigg(\left((-1+\delta)(N-1)^2 - \sqrt{(N-1)^2\left(4\delta+(-1+\delta)^2(N-1)^2\right)}\right)^{T+1} \\
&\quad+\left((-1+\delta)(N-1)^2 + \sqrt{(N-1)^2\left(4\delta+(-1+\delta)^2(N-1)^2\right)}\right)^{T+1}\Bigg)\Bigg].
\end{aligned}$$

The coefficient on the first-order term is

$$\begin{aligned}
&\delta^{T+1+\Delta}\Bigg[\left(-2\delta^{-T} + (1-\delta)\delta^{-(T+1)} + \delta^{-(T+1+\Delta)}(1+\delta)\right)(N-1)^2 \\
&\quad\times\Bigg(\left((-1+\delta)(N-1)^2 - \sqrt{(N-1)^2\left(4\delta+(-1+\delta)^2(N-1)^2\right)}\right)^{T+1}
\end{aligned}$$

$$
\begin{aligned}
&- \left(\left((-1+\delta)(N-1)^2 + \sqrt{(N-1)^2 \left(4\delta + (-1+\delta)^2 (N-1)^2\right)} \right)^{T+1} \right) \\
&- \left(\delta^{-(T+1)} - \delta^{-(T+1+\Delta)} \right) \sqrt{(N-1)^2 \left(4\delta + (-1+\delta)^2 (N-1)^2\right)} \\
&\times \Bigg(\left((-1+\delta)(N-1)^2 - \sqrt{(N-1)^2 \left(4\delta + (-1+\delta)^2 (N-1)^2\right)} \right)^{T+1} \\
&+ \left((-1+\delta)(N-1)^2 + \sqrt{(N-1)^2 \left(4\delta + (-1+\delta)^2 (N-1)^2\right)} \right)^{T+1} \Bigg) \Bigg],
\end{aligned}
$$

and the coefficient on the second-order term is

$$
\begin{aligned}
2\delta^{1+\Delta}(N-1)^2 \Bigg(&\left((-1+\delta)(N-1)^2 - \sqrt{(N-1)^2 \left(4\delta + (-1+\delta)^2 (N-1)^2\right)} \right)^{T+1} \\
&- \left((-1+\delta)(N-1)^2 + \sqrt{(N-1)^2 \left(4\delta + (-1+\delta)^2 (N-1)^2\right)} \right)^{T+1} \Bigg).
\end{aligned}
$$

One can show that the discriminant of the quadratic equation for d is positive, implying that one root is positive and the other is negative. The positive solution describes equilibrium.

An equilibrium with a single no-trade period:

Before proceeding to 24/7 trade, let us prove a result regarding no-trade periods of a single period. In particular, we'll prove the following:

LEMMA 1: *Assume an equilibrium of the conjectured form, 4, with strictly downward sloping demand schedules, $b_k < 0$ for all $k \in \{0, \dots, T\}$, does not exist. Then, if $(N-1)(1-e^{-2rh}) > 1$, there is an equilibrium in which demand schedules are uniformly zero for a single period during the trading day. This equilibrium has no trade in period $T-1$ and also satisfies properties 3 and 4 of Proposition 1 in the other periods.*

It is straightforward to show via numerical examples that the condition $(N-1)(1-e^{-2rh}) > 1$ is not meaningless, i.e., there are parameters for which $1-e^{-2rh} > \frac{1}{N-1} > 1-e^{-rh}$ and only a no-trade period for exactly one period exists.

Note since $\Delta \geq 1$, the condition $(N-1)(1-e^{-2rh}) > 1$ alone implies there is a trade equilibrium at T since the implied c_T is negative. If there were also trade at $T-1$, the oscillation properties (1) and (2) shown above would imply there is trade in all previous periods, a contradiction. Hence, there is no trade at $T-1$.

Now let us show that a_7^{T-1} is large enough for a trade equilibrium to occur in period $T-2$. Or equivalently, we show that the value of c_k necessary for trade to occur is negative.

This will imply, by property (4) above, that there is trade in all prior periods.

Since there's no trade in period $T-1$,

$$a_7^{T-1} = (1-e^{-rh}) + e^{-rh}a_7^T > 1 - e^{-rh}$$

For a trade equilibrium to exist in period $T-2$, we need the necessary value of c_{T-2} to be negative. It is sufficient that

$$1 < (N-1)\left(1-e^{-rh}+e^{-rh}(1-e^{-rh})\right) = (N-1)(1-e^{-2rh}),$$

which holds.

Note there can't be any other equilibrium which satisfies the trembling-hand refinement. This results from our arguments above regarding uniqueness of the refined fixed point problem for a_7. Intuitively, if there were a single other period without trade, it must be that there is trade in periods $T-1, T$, and hence there would be trade in all earlier periods by the oscillation properties.

A.3. Proposition 3: 24/7 Trade

It is straightforward to see that when $\Delta = 0$, solutions to the recursions must be constant. The recursions describing the value function reduce to

$$\begin{aligned}
a_0 &= -\bar{Z}^2c^2\left(\frac{1}{b(N-1)} + e^{-rh}a_4\right) + e^{-rh}a_0 + e^{-rh}a_5\lambda\sigma^2 + e^{-rh}a_6\frac{\lambda\sigma^2}{N} + e^{-rh}a_9\frac{\lambda\sigma^2}{N} \\
a_1 &= \frac{c(c+1)\bar{Z}}{b(N-1)} - \frac{a+c\bar{Z}}{b} \\
a_2 &= -\frac{c\bar{Z}}{N-1} - (1-e^{-rh})c\bar{Z} + e^{-rh}a_2 - e^{-rh}a_7c\bar{Z} \\
a_3 &= \frac{cN\bar{Z}}{N-1} + e^{-rh}a_3 - e^{-rh}a_8c\bar{Z} \\
a_4 &= -\frac{(1-e^{-rh})\gamma_d}{2r}(1+c)^2 + e^{-rh}a_4(1+c)^2 \\
a_5 &= (1-e^{-rh})\frac{f}{2} + \frac{f}{2(N-1)} + e^{-rh}\frac{f(1+c)}{2(N-1)} + e^{-rh}a_5 \\
a_6 &= -\frac{fN}{2(N-1)} - e^{-rh}\frac{f}{2}\left(\frac{N-2}{N-1} - \frac{c}{N-1}\right) + e^{-rh}a_6 \\
a_7 &= \frac{1+c}{N-1}
\end{aligned}$$

$$a_8 = -c + \frac{N-2}{N-1}(1+c)$$
$$a_9 = \frac{cf}{(1+c)(N-1)} + e^{-rh}a_9$$
and the equations describing the trade equilibrium reduce to.
$$b = \frac{r\big(N-2-(N-1)e^{-rh}(1-a_7)\big)}{(N-1)(\gamma_d(e^{-rh}-1)+2re^{-rh}a_4)},$$
$$c = \frac{1}{(N-1)(1+e^{-rh}(a_7-1))} - 1,$$
$$f = \frac{r(1+e^{-rh}(a_7-1))c}{\gamma_d(e^{-rh}-1)+2re^{-rh}a_4},$$
$$a = -\frac{c(N-2)\bar{Z}}{N-1} + b\left(v(e^{-rh}-1) - e^{-rh}a_1 + \frac{c\gamma_d(e^{-rh}-1)\bar{Z}}{r} + 2e^{-rh}c\bar{Z}a_4\right).$$
Therefore,
$$c = \frac{-(N-1)(1-e^{-rh}) + \sqrt{(1-e^{-rh})^2(N-1)^2+4e^{-rh}}}{2e^{-rh}} - 1.$$
Given c, we can solve for a_7 and a_4. This yields solutions for b, f, a, and the remaining recursions.

B. *Information Problem*

This appendix characterizes the solution of the model when agents have heterogeneous asset values. Recall S^j is each trader's total signal (sum of past signals). s^j is each trader's modified signal. Write their expectation of the dividend as
$$w^j + B_1 S^j + B_2 \sum_{i \neq j} \left(w^i + AS^i\right),$$
for some constants B_1, B_2, A. Consistency of the learning problem requires $B_1 = A$. See Du and Zhu (2017b) for details. Recall the variance of private value shocks is σ^2, of dividend shocks is σ_D^2, and of signal shocks is σ_ϵ^2. Then, Du and Zhu (2017b) Lemma 1 gives the conditional expectation of v given w^j, S^j, and $\sum_{i \neq j}(w^i + AS^i)$ is
$$w^j + \frac{1/(A^2\sigma_\epsilon^2)}{1/(A^2\sigma_D^2) + 1/(A^2\sigma_\epsilon^2) + (n-1)/(A^2\sigma_\epsilon^2+\sigma^2)} S^j +$$

$$\frac{1/(A^2\sigma_\epsilon^2+\sigma^2)}{1/(A^2\sigma_D^2)+1/(A^2\sigma_\epsilon^2)+(n-1)/(A^2\sigma_\epsilon^2+\sigma^2)}\frac{1}{A}\sum_{i\neq j}\left(w^i+AS^i\right).$$

B_1 is defined in terms of A by the above. A solves the equation $A = B_1$, and B_2 is then given as a function of A.

Define

$$s^j = \frac{1}{\alpha}(w^j + B_1 S^j),$$

where

$$\alpha = \frac{A^2\sigma_\epsilon^2+\sigma^2}{NA^2\sigma_\epsilon^2+\sigma^2}.$$

Then, the conditional expectation of v is given by

$$\alpha s^j + \frac{1-\alpha}{N-1}s^{-j} = \frac{N\alpha-1}{N-1}s^j + \frac{N(1-\alpha)}{N-1}\bar{s}.$$

Guess that the value function is linear-quadratic:

$$V_k(z^j,\bar{Z},s^j,\bar{s}) = a_0^k + a_1^k z^j + a_2^k s^j + a_3^k\bar{s} + a_4^k(z^j)^2 + a_5^k(s^j)^2 + a_6^k(\bar{s})^2 + a_7^k z^j s^j + a_8^k z^j\bar{s} + a_9^k s^j\bar{s}.$$

$\sigma^2 = \frac{1}{\alpha^2}(\sigma^2 + A^2(\sigma_D^2+\sigma_\epsilon^2))$ is variance of the shock to s^j, and $\sigma_N^2 = \frac{1}{\alpha^2}(\sigma^2/N + A^2(\sigma_D^2+\sigma_\epsilon^2/N))$ is the variance of the shocks to $\bar{s}$. The Bellman equation for every period, except the last, is

$$\begin{aligned}
V_k(z^j,s^j,\bar{s}) = \max_{D^j}\Bigg\{ & -D^j p_t^* + (1-e^{-rh})(z^j+D^j)\left(\frac{N\alpha-1}{N-1}s^j + \frac{N(1-\alpha)}{N-1}\bar{s}\right) \\
& -\frac{(1-e^{-rh})\gamma_d}{2r}(z^j+D^j)^2 + e^{-rh}\Big[a_0^{t+1} + a_1^{k+1}(z^j+D^j) + a_2^{k+1}s^j + a_3^{k+1}\bar{s} \\
& a_4^{k+1}(z^j+D^j)^2 + a_5^{k+1}((s^j)^2+\lambda\sigma^2) + a_6^{k+1}(\bar{s}^2+\lambda\sigma_N^2) \\
& +a_7^{k+1}(z^j+D^j)s^j + a_8^{k+1}(z^j+D^j)\bar{s} + a_9^{k+1}(s^j\bar{s}_t+\lambda\sigma_N^2)\Big]\Bigg\},
\end{aligned}$$

and it is similar in the last period. The FOC for optimal demand in the first T periods is then

$$\begin{aligned}
0 = -p_t^* - \lambda_k D^j + (1-e^{-rh})\left(\frac{N\alpha-1}{N-1}s^j + \frac{N(1-\alpha)}{N-1}\bar{s}\right) \\
-\frac{(1-e^{-rh})\gamma_d}{r}(z^j+D^j) + e^{-rh}[a_1^{k+1} + 2a_4^{k+1}(z^j+D^j) + a_7^{k+1}s^j + a_8^{k+1}\bar{s}],
\end{aligned}$$

where $\lambda_k := \frac{\partial p_t}{\partial D_k^j}$. Assume

$$D_k^j = a_k + b_k p_t + c_k z^j + f_k s^j.$$

The equilibrium price is

$$p_t = -\frac{a_k + c_k\bar{Z} + f_k\bar{s}_t}{b_k}.$$

The FOC implies

$$\begin{aligned}
&\frac{a_k + c_k\bar{Z} + f_k\bar{s}}{b_k} + \frac{1}{b_k(N-1)}(c_k(z^j - \bar{Z}) + f_k(s^j - \bar{s})) \\
&+ (1-e^{-rh})\left(\frac{N\alpha - 1}{N-1}s^j + \frac{N(1-\alpha)}{N-1}\bar{s}\right) - \frac{(1-e^{-rh})\gamma_d}{r}((1+c_k)z^j - c_k\bar{Z} + f_k(s^j - \bar{s})) \\
&\qquad + e^{-rh}\left[a_1^{k+1} + 2a_4^{k+1}((1+c_k)z^j - c_k\bar{Z} + f_k(s^j - \bar{s})) + a_7^{k+1}s^j + a_8^{k+1}\bar{s}\right] = 0.
\end{aligned}$$

Then

$$b = -\frac{e^{hr}\left(-a_8 + a_7(-2+N) + (-1+e^{hr})(-2+\alpha N)\right)r}{((-1+a_7+a_8+e^{hr})(-1+N)\left((-1+e^{hr})\gamma - 2a_4r)\right)}$$

$$c = \frac{a_8 - a_7(-2+N) - (-1+e^{hr})(-2+\alpha N)}{a_7(-1+N) + (-1+e^{hr})(-1+\alpha N)}$$

$$f = -\frac{\left(-a_8 + a_7(-2+N) + (-1+e^{hr})(-2+\alpha N)\right)r}{(-1+N)\left(\gamma - e^{hr}\gamma + 2a_4r\right)}$$

Returning to the Bellman equation, we have

$$\begin{aligned}
V_k = &(c_k(z^j - \bar{Z}) + f_k(s^j - \bar{s}))(\frac{a_k}{b_k} + \frac{c_k}{b_k}\bar{Z} + \frac{f_k}{b_k}\bar{s}) \\
&+ (1-e^{-rh})((1+c_k)z^j - c_k\bar{Z} + f_k(s^j - \bar{s}))\left(\frac{N\alpha - 1}{N-1}s^j + \frac{N(1-\alpha)}{N-1}\bar{s}\right) \\
&- \frac{(1-e^{-rh})\gamma_d}{2r}(((1+c_k)z^j - c_k\bar{Z} + f_k(s^j - \bar{s})))^2 \\
&+ e^{-rh}\left[a_0^{t+1} + a_1^{k+1}((1+c_k)z^j - c_k\bar{Z} + f_k(s^j - \bar{s})) + a_2^{k+1}s^j + a_3^{k+1}\bar{s}\right. \\
&a_4^{k+1}((1+c_k)z^j - c_k\bar{Z} + f_k(s^j - \bar{s}))^2 + a_5^{k+1}((s^j)^2 + \lambda\sigma^2) + a_6^{k+1}(\bar{s}^2 + \lambda\sigma_N^2) \\
&+a_7^{k+1}((1+c_k)z^j - c_k\bar{Z} + f_k(s^j - \bar{s}))s^j \\
&\left.+a_8^{k+1}((1+c_k)z^j - c_k\bar{Z} + f_k(s^j - \bar{s}))\bar{s} + a_9^{k+1}(s^j\bar{s} + \lambda\sigma_N^2)\right],
\end{aligned}$$

which yields recursions as before.

Internet Appendix of: *Is 24/7 Trading Better?*

Appendix IA.1 provides more details of the calibration and the data used. Appendix IA.2 studies welfare when parameters vary between night and day. Appendix IA.3 quantifies welfare relative to perfectly efficient trade. Appendix IA.4 solves the continuous trade model, computes the expected volume in that model, studies an exchange's problem, and shows the convergence of the discrete trade model to that solution. Last, Appendix IA.5 provides some simplifications of the recursions provided in the Appendix.

IA.1 Calibration Details

We calibrate our model to some large equity exchanges to study counterfactual values, such as welfare or volume, when the length of the trading day changes. To do this, we need estimates of a few parameters per exchange. The optimal length of a closure, Δ^*, depends on N and $\frac{\sigma_d}{\sigma_n}$. Therefore, we need at least two linearly independent empirical moments from each exchange to identify these parameters. Due to the availability of Trade and Quote (TAQ) data, we choose to use the fraction of total daily volume in certain time intervals. Specifically, we compute the total volume in 2023 between 9:30 a.m. and 4:00 p.m. per exchange and the total volume in each 30-minute interval per exchange. From this, we can compute the average fraction of daily volume from 9:30 to 12:30 and 1:00 to 4:00. We leave out the interval 12:30 to 1:00 so that the moments are not linear combinations of each other. We then compute the corresponding measure implied by our model. Mathematically, this is

$$\frac{\mathbb{E}\left[Volume_{[x,x+\frac{3}{24})}\right]}{\mathbb{E}\left[Volume_{[0,1-\Delta)}\right]} = \frac{\int_x^{x+\frac{3}{24}} E\left[\sum_{i=1}^N |D_t^i|\right] dt}{\int_0^{1-\Delta} E\left[\sum_{i=1}^N |D_t^t|\right] dt},$$

where $x = 0$ is the start of trading, 9:30 a.m. Note that the exchanges we focus on all trade for 6.5 hours a day, so $\Delta = \frac{17.5}{24}$. Section IA.4.1 details the calculation of expected instantaneous volume. Note the above formula abuses notation, since trades at the end-of-day session are discrete quantities, not flows. These discrete trades can be thought of as Dirac delta functions in the integral above. Table A.1 lists the empirical moments, the model implied moments, and the calibrated parameters per exchange, which are fit by the method of moments. Our model fits the data well.

Table A.1
Empirical and Calibrated Moments

This table compares the fraction of daily volume per exchange from 9:30-12:30 and 1:00-4:00 to that from the calibrated model, as well as the calibrated parameters. $\widehat{N}$ denotes the estimated size of the market, and $\widehat{\frac{\sigma_d}{\sigma_n}}$ is the relative instantaneous volatilities during the day and night. We assume $r = 10\%$, $v = 0$, and $z_0^i = 0$ for all calibrations.

Exchange	Current Length of Night (Δ)	Empirical Volume 9:30-12:30	Empirical Volume 1:00-4:00	Calibrated Volume 9:30-12:30	Calibrated Volume 1:00-4:00	$\widehat{N}$	$\widehat{\frac{\sigma_d}{\sigma_n}}$
NYSE	72.9%	49.9%	46.0%	49.9%	44.8%	208	1.28
Nasdaq	72.9%	50.2%	45.1%	50.2%	44.5%	325	1.32
NYSE Arca	72.9%	54.5%	40.6%	54.5%	40.4%	303	1.23
CBOE EDGX	72.9%	54.8%	40.0%	54.8%	40.0%	191	0.87

IA.2 Welfare when Night Characteristics Differ From the Day

Throughout, we have assumed that marginal holding costs and the private value shock process have been the same whether the market is open or closed. However, this is unlikely to be true. In this Appendix, we look at the welfare gain (or loss) of a short market closure of one hour versus 24/7 trading when holding costs or shock magnitudes differ between night and day. We choose to focus on the case of a one-hour closure as this is a common closure length proposed for extending hours by the NYSE, Nasdaq, CBOE, and 24X.

Figure B.1 plots an example. The blue dotted line varies the volatility of shocks to private values at night while holding the total volatility in a day fixed. Mathematically, $\sigma_d = \sqrt{\frac{\sigma_T^2 - \Delta\sigma_n^2}{1-\Delta}}$. This choice ensures potential gains from trade, which are larger when there are more shocks to private values, are not a function of the length of the closure. When volatility at night is less than the total volatility, there is an increase in welfare due to the hour-long closure, and welfare decreases when the night is more volatile. The solid red line plots the change in welfare as a function of the change in the marginal holding cost from day to night. As it becomes cheaper to hold inventory overnight when $\gamma_d > \gamma_n$, there are large welfare gains. When $\gamma_d < \gamma_n$, the hour-long closure rapidly hurts welfare relative to having the market open 24/7.

IA.3 The Cost of Imperfect Competition for Differing Closure Lengths

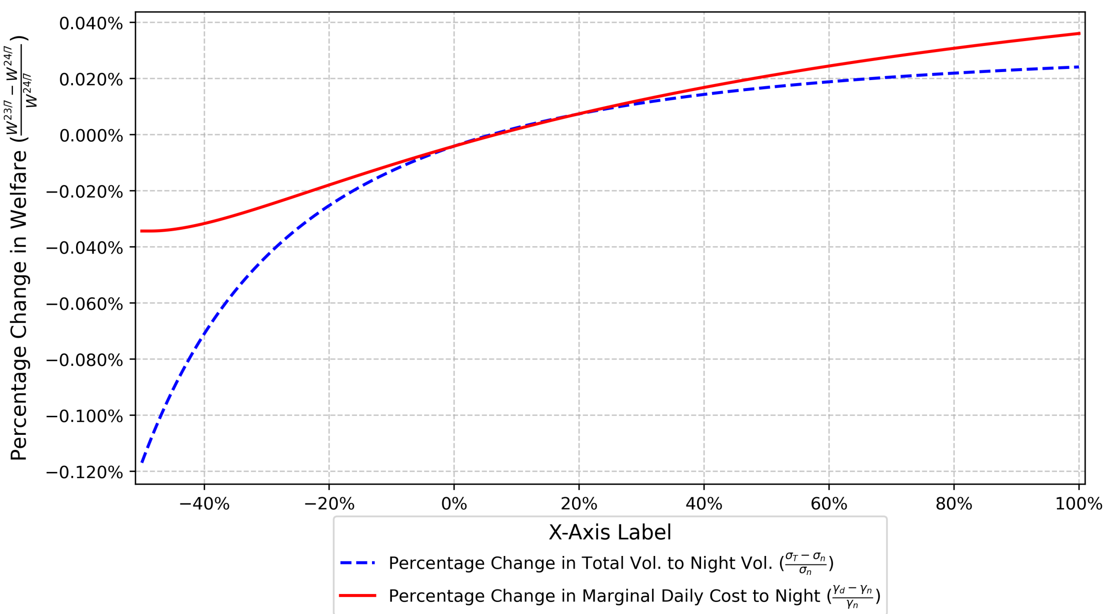

Figure B.1. Welfare Change Under Heterogeneity From Day to Night
Above is the percent change between welfare under a market closure of one hour and welfare under 24/7 trade as we vary the marginal holding cost or volatility of the shocks between night to day. The dotted blue line plots the welfare change as a function of marginal holding cost during the day compared to that of the night. The solid red line plots the welfare change as a function of total volatility, $\sigma_T^2 = (1-\Delta)\sigma_d^2 + \Delta\sigma_n^2$, relative to volatility at night, where σ_d solves that equation. Both plots use $\Delta = 1/24$, $r = 10\%$, $\lambda = 10$, $N = 10$, and σ and γ equal 1 unless specified to be different.

Throughout, we have focused on comparing welfare under a market structure with 24/7 trade and with a daily closure, ignoring the cost of each relative to the first-best allocation. The first-best allocation would be achieved if there were perfect competition and if the trade occurred continuously throughout the day. In this setting, no trader ever holds any undesired inventory. Making comparisons relative to the first-best allocation allows us to better quantify the costs and benefits of market closure.

Figure C.1 plots the percentage of welfare loss of different market designs relative to the first-best (efficient) allocations. Panel A is for a small market, and Panel B is for a large market. The solid red line is the welfare loss of a market design with 24/7 trade relative to efficient welfare. The dashed blue line is the welfare loss of a market design that is closed for

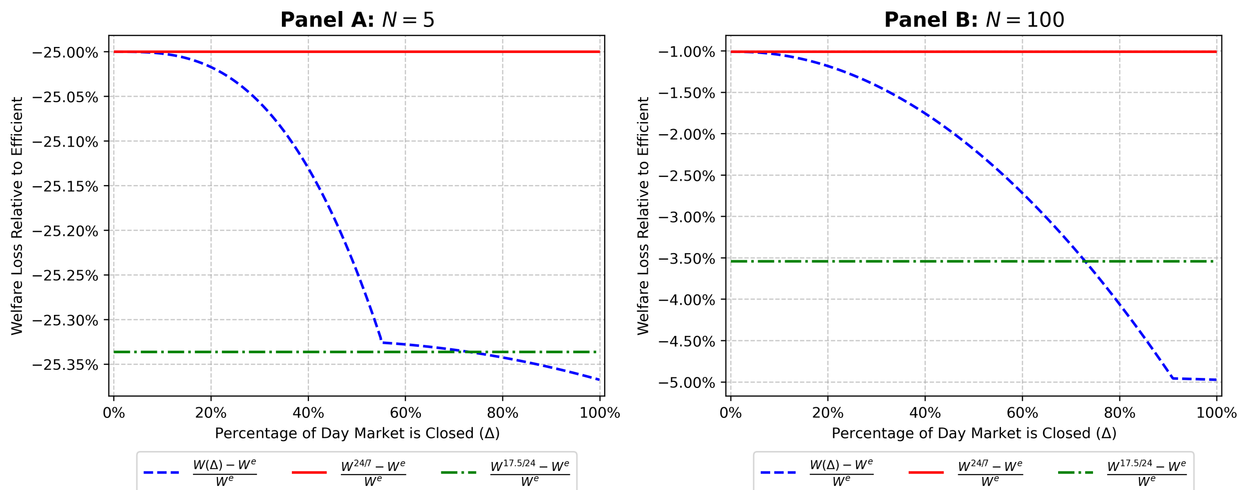

Figure C.1. Welfare Loss Relative to Efficient Benchmark
We plot the percent welfare loss under different market designs relative to the first best (efficient) welfare. Panel A plots this loss for a small market, and Panel B plots this loss for a large market. The solid red line is the welfare loss of a market design that is open 24/7 relative to the efficient welfare. The dashed blue line is the welfare loss of a market design that is closed for Δ periods a day relative to the efficient welfare. The dashed-and-dotted green line is the welfare loss of a market design that is closed for 17.5 hours a day, such as many equity exchanges, relative to efficient welfare. Both plots use $r = 10\%$, and $\lambda = 10$.

Δ periods a day relative to efficient welfare. The dashed-and-dotted green line is the welfare loss of a market design that is closed for 17.5 hours a day, such as many equity exchanges, relative to efficient welfare.

As before, the 24/7 market is better for traders than a market with a closure. However, in Panel A, the welfare loss due to closure is very small, less than 0.40%, relative to the overall welfare loss of imperfect competition, $\approx$ 25%. Take, for example, when $\Delta \approx 50\%$. The welfare cost is only an extra 0.30% worse than 24/7 trading, despite only allowing trade for 50% of the day. The endogenous response by traders and coordination of liquidity at the end of the day offset the majority of the extra costs incurred due to the inability to trade at night. The current equity market structure, which involves trading for 6.5 hours a day, is associated with approximately 0.33% extra loss in welfare relative to the efficient benchmark.

In Panel B, when the market is larger, closure becomes relatively more costly. Now, trading for 6.5 hours a day has 3.5 times the welfare loss relative to the efficient benchmark. In this larger market, the costs relative to the efficient benchmark are significantly lower, as the friction from imperfect competition is less important. Therefore, long closures are fairly costly in these larger and more liquid markets, whose liquidity wouldn't endogenously deteriorate too much if trading hours were extended. It is worth noting that these results assume constant volatility and holding costs across the day and night.

IA.4 Continuous Trade Model

The continuous time model assumes there is trade from time 0 to $1-\Delta-\epsilon$, a no-trade period from $1-\Delta-\epsilon$ to $1-\Delta$, a discrete trade session at $1-\Delta$ and then an overnight period from $1-\Delta$ to 1. If D_t^i is the demand allocated at time t, we have $dz^i = D_t^i dt$ during the continuous trade sessions. We will allow various parameters to have separate d and n subscripts to show they can differ between $[0, 1-\Delta)$ and $[1-\Delta, 1]$.

We begin by solving the model for continuous trading sessions between 0 and $1-\Delta-\epsilon$. We conjecture that the other $N-1$ traders submit demand schedules given by Equation 4. Trade is modeled by a uniform price double auction where the price is the solution to Equation 1. Therefore, the equilibrium price is

$$p_t^* = -\frac{a(t) + c(t)\bar{Z} + f(t)\bar{W}_t}{b(t)}.$$

Note that a, b, and c are functions of t which are not explicitly defined between $1-\Delta-\epsilon$ and $1-\Delta$. Given the equilibrium price, the demand schedule evaluated at the equilibrium price is

$$D_t^i = c(t)\big(z_t^i - \bar{Z}\big) + f(t)(w_t^i - \bar{W}_t).$$

Finally, conjecture that the day value function takes the following linear-quadratic form

$$\begin{aligned} J^d(t, z^i, w^i, \bar{W}) = \alpha_0(t) + \alpha_1(t)z^i + \alpha_2(t)w^i + \alpha_3(t)\bar{W} + \alpha_4(t)(z^i)^2 + \alpha_5(t)(w^i)^2 + \alpha_6(t)(\bar{W})^2 \\ + \alpha_7(t)z^i w^i + \alpha_8(t)z^i\bar{W} + \alpha_9(t)w^i\bar{W}. \end{aligned}$$

Recall that traders rationally anticipate how their demand affects their trade price. Therefore, when trader i chooses demand d^i, they face the residual demand curve that, by market clearing, implies they face the price $\Phi(t, d^i, z^i, W^{-i})$, defined in equation 5. Therefore, the Hamilton-Jacobi-Bellman equation is

$$rJ^d = \max_{d^i} \big\{ J_t^d + rz^i(v + w^i) - \Phi(t, d^i, z^i, W^{-i})d^i - \frac{\gamma_d}{2}(z^i)^2 + J_{z^i}^d d^i$$
$$+ \lambda_d E_t\big[J^d(t, z^i, w^i + \xi^i, \bar{W} + \bar{\xi}) - J^d(t, z^i, w^i, \bar{W})\big]\big\},$$

where $\xi_i \overset{iid}{\sim} N(0, \sigma_d^2)$. First, we will solve for the equations that define the α functions, and then we will add in the optimality of demand constraints. Plugging the conjectured day value function into the HJB equation, as well as the equilibrium price and demand schedule, we get

$$\begin{aligned}
&r(\alpha_0(t) + \alpha_1(t)z^i + \alpha_2(t)w^i + \alpha_3(t)\bar{W} + \alpha_4(t)(z^i)^2 + \alpha_5(t)(w^i)^2 \\
&\qquad + \alpha_6(t)\bar{W}^2 + \alpha_7(t)z^i w^i + \alpha_8(t)z^i\bar{W} + \alpha_9(t)w^i\bar{W}) \\
&= \alpha_0'(t) + \alpha_1'(t)z^i + \alpha_2'(t)w^i + \alpha_3'(t)\bar{W} + \alpha_4'(t)(z^i)^2 + \alpha_5'(t)(w^i)^2 \\
&+ \alpha_6'(t)\bar{W}^2 + \alpha_7'(t)z^i w^i + \alpha_8'(t)z^i\bar{W} + \alpha_9'(t)w^i\bar{W} + z^i r(v + w^i) \\
&- \frac{1}{b(t)(N-1)}(c(t)(z^i - \bar{Z}) + f(t)(w^i - \bar{W}))^2 - \frac{\gamma_d}{2}(z^i)^2 + \lambda_d(\alpha_5(t)\sigma_d^2 + \alpha_6(t)\frac{\sigma_d^2}{N} + \alpha_9(t)\frac{\sigma_d^2}{n}).
\end{aligned}$$

By matching coefficients, we get that

$$\begin{aligned}
r\alpha_0(t) &= \alpha_0'(t) - \frac{c(t)^2\bar{Z}^2}{b(t)(N-1)} + \lambda_d(\alpha_5(t)\sigma_d^2 + \alpha_6(t)\frac{\sigma_d^2}{N} + \alpha_9(t)\frac{\sigma_d^2}{N}) \\
r\alpha_1(t) &= \alpha_1'(t) + rv + \frac{2}{b(t)(N-1)}c(t)\bar{Z} \\
r\alpha_2(t) &= \alpha_2'(t) + \frac{2}{b(t)(N-1)}f(t)\bar{Z} \\
r\alpha_3(t) &= \alpha_3'(t) - \frac{2}{b(t)(N-1)}f(t)\bar{Z} \\
r\alpha_4(t) &= \alpha_4'(t) - \frac{\gamma_d}{2} - \frac{c(t)^2}{b(t)(N-1)} \\
r\alpha_5(t) &= \alpha_5'(t) - \frac{f(t)^2}{b(t)(N-1)} \\
r\alpha_6(t) &= \alpha_6'(t) - \frac{f(t)^2}{b(t)(N-1)} \\
r\alpha_7(t) &= \alpha_7'(t) + r - \frac{2f(t)c(t)}{b(t)(N-1)} \\
r\alpha_8(t) &= \alpha_8'(t) + \frac{2f(t)c(t)}{b(t)(N-1)}
\end{aligned}$$

$$r\alpha_9(t) = \alpha_9'(t) + \frac{2f(t)^2}{b(t)(N-1)}.$$

To get the optimality of demand equations, we take the first-order condition of the right side of the HJB equation with respect to d^i. This yields the equation

$$-\Phi - \Phi_{d^i} d^i + J^d_{z^i} = 0.$$

Plugging in the equilibrium expressions for Φ and d^i, we are left with the equations

$$\frac{a(t) + c(t)\bar{Z} + f(t)\bar{W}}{b(t)} + \frac{1}{b(t)(N-1)}(c(t)(z^i - \bar{Z}) + f(t)(w^i - \bar{W})) \\ + \alpha_1(t) + 2\alpha_4(t)z^i + \alpha_7(t)w^i + \alpha_8(t)\bar{W} = 0.$$

Matching coefficients in the above equation gives us four equations that must be satisfied for demand to be optimal:

$$\frac{a(t) + c(t)\bar{Z}}{b(t)} - \frac{1}{b(t)(N-1)}c(t)\bar{Z} + \alpha_1(t) = 0,$$
$$\frac{c(t)}{b(t)(N-1)} + 2\alpha_4(t) = 0,$$
$$\frac{f(t)}{b(t)(N-1)} + \alpha_7(t) = 0,$$
$$\frac{f(t)}{b(t)} - \frac{f(t)}{b(t)(N-1)} + \alpha_8(t) = 0.$$

From optimality of demand, $\alpha_8(t) = -\frac{(N-2)f(t)}{(N-1)b(t)} = (N-2)\alpha_7(t)$. Summing the equations for $\alpha_7(t)$, and $\alpha_8(t)$ we have

$$\alpha_7(t) = A_7 e^{rt} + \frac{1}{N-1}.$$

Plugging this back into the equation for $\alpha_7(t)$,

$$rA_7 e^{\lambda t} + \frac{\lambda}{N-1} = rA_7 e^{rt} + r + 2c\alpha_7(t),$$

so $c(t) = \frac{-r(N-2)}{2(A_7(N-1)e^{rt}+1)}$.

Assume A_4 through A_9 are 0, so α_4 through α_9 are constant too. We will argue later that this conjecture is satisfied in equilibrium. Then $c = -\frac{r(N-2)}{2}$, and $\alpha_7 = \frac{1}{N-1}$. The equation for α_4 becomes

$$r\alpha_4 = -\frac{\gamma_d}{2} - \alpha_4 r(N-2),$$

so $\alpha_4 = -\frac{2\gamma_d}{r(N-1)}$. This implies

$$b(t) = -\frac{c(t)}{2\alpha_4(N-1)} = -\frac{r^2(N-2)}{2\gamma_d}, \quad \text{and } f(t) = -\alpha_7(N-1)b(t) = \frac{r^2(N-2)}{2\gamma_d}.$$

So, $b(t)$, $c(t)$, and $f(t)$ are all constant between time 0 and $1-\Delta-\epsilon$. Solving the differential equations for the α's, we get

$$\begin{aligned}
\alpha_0(t) &= \frac{\gamma_d(N-2)\bar{Z}^2}{2r(N-1)} - e^{rt}\int_0^t e^{-rs}\lambda_d\left(\alpha_5(s)\sigma_d^2 + \alpha_6(s)\frac{\sigma_d^2}{N} + \alpha_9(s)\frac{\sigma_d^2}{N}\right)ds + A_0 e^{rt}\\
\alpha_1(t) &= A_1 e^{rt} + v + \frac{4\gamma_d\bar{Z}}{r^2(N-1)}\\
\alpha_2(t) &= A_2 e^{rt} - \frac{2\bar{Z}}{r(N-1)}\\
\alpha_3(t) &= A_3 e^{rt} + \frac{2\bar{Z}}{r(N-1)}\\
\alpha_4 &= -\frac{\gamma_d}{2r(N-1)}\\
\alpha_5 &= \frac{r(N-2)}{2\gamma_d(N-1)}\\
\alpha_6 &= \frac{r(N-2)}{2\gamma_d(N-1)}\\
\alpha_7 &= \frac{1}{N-1}\\
\alpha_8 &= \frac{N-2}{N-1}\\
\alpha_9 &= -\frac{r(N-2)}{\gamma_d(N-1)}
\end{aligned}$$

Plugging in α_5, α_6, and α_9 into α_0 and simplifying gives

$$\alpha_0(t) = \frac{\gamma_d(N-2)\bar{Z}^2}{2r(N-1)} - \lambda_d\sigma_d^2\frac{(N-2)}{2\gamma_d N}\left(e^{rt}-1\right) + A_0 e^{rt}.$$

After the continuous trade sessions, there is a no-trade period of length ϵ where no trade occurs, and then there is a closing auction at time $1-\Delta$. Therefore, the value function right before the no-trade period is

$$\begin{aligned}
J^d(t = 1-\Delta-\epsilon, z^i, w^i, \bar{W}) =& (1-e^{-r\epsilon})\left(z^i(v+w^i) - \frac{\gamma_d}{2r}\left(z^i\right)^2\right)\\
&+ e^{-r\epsilon}E_{1-\Delta-\epsilon}\left[J^d(t = 1-\Delta^-, z^i, w^i_{t-\Delta}, \bar{W}_{t-\Delta})\right].
\end{aligned}$$

Now, we move on to the discrete auction at the close, $t = 1 - \Delta$. Recall that traders rationally anticipate how their demand affects their trade price. Therefore, when trader i chooses demand d^i, they face the residual demand curve that, by market clearing, implies they face the price $\Phi(t, d^i, z^i, W^{-i})$, defined in equation 5. Therefore, the value function at $1 - \Delta^-$ satisfies

$$J^d(t = 1 - \Delta^-, z^i, w^i, \bar{W}) = \max_{d^i} \{J^n(t = 1 - \Delta^+, z^i + d^i, w^i, \bar{W}) - \Phi(1 - \Delta, d^i, z^i, W^{-i}) d^i\},$$

where J^n describes the value function overnight. We conjecture

$$\begin{aligned} J^n(t, z^i, w^i, \bar{W}) = \beta_0(t) + \beta_1(t) z^i + \beta_2(t) w^i + \beta_3(t) \bar{W} + \beta_4(t)(z^i)^2 + \beta_5(t)(w^i)^2 + \beta_6(t)(\bar{W})^2 \\ + \beta_7(t) z^i w^i + \beta_8(t) z^i \bar{W} + \beta_9(t) w^i \bar{W}. \end{aligned}$$

To get the optimality of demand equations, we take the first-order condition of the right side of the equation for $J^d(t = 1 - \Delta^-, z^i, w^i, \bar{W})$ with respect to d^i. This yields

$$-\Phi - \Phi_{d^i} d^i + J^n_{d^i} = 0.$$

Plugging in the equilibrium expressions for Φ and d^i, we are left with

$$\begin{aligned} \frac{a(1-\Delta) + c(1-\Delta)\bar{Z} + f(1-\Delta)\bar{W}}{b(1-\Delta)} + \frac{1}{b(1-\Delta)(N-1)} d^i + \\ \beta_1(1-\Delta) + 2\beta_4(1-\Delta)(z^i + d^i) + \beta_7(1-\Delta) w^i + \beta_8(1-\Delta)\bar{W} = 0. \end{aligned}$$

First, plug in the equilibrium demand for d^i, which gives

$$\begin{aligned} \frac{a(1-\Delta) + c(1-\Delta)\bar{Z} + f(1-\Delta)\bar{W}}{b(1-\Delta)} + \frac{1}{b(1-\Delta)(N-1)} (c(1-\Delta)(z^i - \bar{Z}) + f(1-\Delta)(w^i - \bar{W})) \\ + \beta_1(1-\Delta) + 2\beta_4(1-\Delta)((1 + c(1-\Delta)) z^i - \bar{Z} c(1-\Delta) + f(1-\Delta)(w^i - \bar{W})) \\ + \beta_7(1-\Delta) w^i + \beta_8(1-\Delta)\bar{W} = 0. \end{aligned}$$

Matching coefficients in the above equation gives us four equations that must be satisfied for demand at the closing auction to be optimal,

$$\begin{aligned} \frac{a(1-\Delta) + c(1-\Delta)\bar{Z}}{b(1-\Delta)} - \frac{1}{b(1-\Delta)(N-1)} c(1-\Delta)\bar{Z} + \beta_1(1-\Delta) - 2\beta_4(1-\Delta) c(1-\Delta)\bar{Z} = 0, \\ \frac{c(1-\Delta)}{b(1-\Delta)(N-1)} + 2\beta_4(1-\Delta)(1 + c(1-\Delta)) = 0, \\ \frac{f(1-\Delta)}{b(1-\Delta)(N-1)} + 2\beta_4(1-\Delta) f(1-\Delta) + \beta_7(1-\Delta) = 0, \end{aligned}$$

$$\frac{f(1-\Delta)}{b(1-\Delta)} - \frac{f(1-\Delta)}{b(1-\Delta)(N-1)} - 2\beta_4(1-\Delta)f(1-\Delta) + \beta_8(1-\Delta) = 0.$$

Now, we move on to the solution of the value function at night. The HJB equation is

$$\begin{aligned} r\big(&\beta_0(t) + \beta_1(t)z^i + \beta_2(t)w^j + \beta_3(t)\bar{W} + \beta_4(t)(z_t^i)^2 + \beta_5(t)(z^j)^2 \\ &+ \beta_6(t)\bar{W}^2 + \beta_7(t)z^jw^j + \beta_8(t)z^j\bar{W} + \beta_9(t)w^j\bar{W}\big) \\ &= \beta_0'(t) + \beta_1'(t)z^i + \beta_2'(t)w^j + \beta_3'(t)\bar{W} + \beta_4'(t)(z^i)^2 + \beta_5'(t)(w^i)^2 \\ &\quad + \beta_6'(t)\bar{W}^2 + \beta_7'(t)z^jw^j + \beta_8'(t)z^j\bar{W} + \beta_9'(t)w^j\bar{W} \\ &\quad + rz_t^i(v + w^j) - \frac{\gamma_n}{2}(z_t^i)^2 + \lambda_n(\beta_5(t)\sigma_n^2 + \beta_6(t)\frac{\sigma_n^2}{N} + \beta_9(t)\frac{\sigma_n^2}{N}). \end{aligned}$$

By matching coefficients, we get

$$\begin{aligned} r\beta_0(t) &= \beta_0'(t) + \lambda_n(\beta_5(t)\sigma_n^2 + \beta_6(t)\frac{\sigma_n^2}{N} + \beta_9(t)\frac{\sigma_n^2}{N}) \\ r\beta_1(t) &= \beta_1'(t) + rv \\ r\beta_2(t) &= \beta_2'(t) \\ r\beta_3(t) &= \beta_3'(t) \\ r\beta_4(t) &= \beta_4'(t) - \frac{\gamma_n}{2} \\ r\beta_5(t) &= \beta_5'(t) \\ r\beta_6(t) &= \beta_6'(t) \\ r\beta_7(t) &= \beta_7'(t) + r \\ r\beta_8(t) &= \beta_8'(t) \\ r\beta_9(t) &= \beta_9'(t) \end{aligned}$$

Solving the above ODEs yields the following equations

$$\begin{aligned} \beta_0(t) &= -e^{rt}\int_{1-\Delta}^{t} \lambda_n e^{-rs}\left(\beta_5(s)\sigma_n^2 + \beta_6(s)\frac{\sigma_n^2}{N} + \beta_9(s)\frac{\sigma_n^2}{N}\right)ds + B_0e^{rt} \\ \beta_1(t) &= B_1e^{rt} + v \\ \beta_2(t) &= B_2e^{rt} \\ \beta_3(t) &= B_3e^{rt} \\ \beta_4(t) &= -\frac{\gamma_n}{2r} + B_4e^{rt} \\ \beta_5(t) &= B_5e^{rt} \end{aligned}$$

$$
\begin{aligned}
\beta_6(t) &= B_6 e^{rt} \\
\beta_7(t) &= 1 + B_7 e^{rt} \\
\beta_8(t) &= B_8 e^{rt} \\
\beta_9(t) &= B_9 e^{rt}
\end{aligned}
$$

Note that $\beta_0(t)$ can be simplified to

$$\beta_0(t) = e^{rt}\bigg(B_0 - \lambda_n \sigma_n^2 \big(B_5 + \frac{B_6 + B_9}{N}\big)\big(t - (1-\Delta)\big)\bigg).$$

All that is left now is to solve the constants in the solutions for the α's and β's using boundary value matching conditions and periodicity. The two boundary conditions are

$$
\begin{aligned}
J^d(t = 1-\Delta^-, z^i, w^i, \bar{W}) = J^n(t = 1-\Delta, z^i + c(1-\Delta)(z^i - \bar{Z}) + f(1-\Delta)(w^i - \bar{W}), w^i, \bar{W}) \\
- \Phi(1-\Delta, c(1-\Delta)(z^i - \bar{Z}) + f(1-\Delta)(w^i - \bar{W}), z^i W^{-i}) d^i.
\end{aligned}
$$

and $\lim\limits_{t \to 1^-} J^n(t, z^i, w^i, \bar{W}) = \lim\limits_{t \to 1^-} \mathbb{E}_t \left[J^d(t = 0, z^i, w^i, \bar{W})\right]$.

The first boundary condition is more involved. After the closing auction, the night value function is actually

$$
\begin{aligned}
&J^n(1-\Delta, z^i + c(1-\Delta)\big(z^i - Z\big) + f(1-\Delta)(w^i - \bar{W}), w^i, \bar{W}) \\
&\quad = \beta_0(1-\Delta) + \beta_1(1-\Delta)\bigg(z_t^i + c(1-\Delta)\big(z_t^i - Z_t\big) + f(1-\Delta)(w^i - \bar{W})\bigg) + \\
&\qquad \beta_2(1-\Delta) w^i + \beta_3(1-\Delta)\bar{W} \\
&+ \beta_4(1-\Delta)\bigg(z_i + c(1-\Delta)\big(z^i - Z\big) + f(1-\Delta)(w^i - \bar{W})\bigg)^2 + \beta_5(1-\Delta)(w^i)^2 + \beta_6(1-\Delta)\bar{W}^2 \\
&+ \bigg(z_t^i + c(1-\Delta)\big(z_t^i - Z_t\big) + f(1-\Delta)(w^i - \bar{W})\bigg)\big(\beta_7(1-\Delta) w^i + \beta_8(1-\Delta)\bar{W}\big) + \beta_9(1-\Delta) w^i \bar{W}.
\end{aligned}
$$

Combining like terms gives and subtracting off the costs of the trade gives the value at $1-\Delta^-$:

$$
\begin{aligned}
&J^n(1-\Delta, z^i + c(1-\Delta)\big(z^i - Z\big) + f(1-\Delta)(w^i - \bar{W}), w^i, \bar{W}) - \Phi(1-\Delta, d^i, z^i, W^{-i}) d^i \\
&= \beta_0(1-\Delta) - \beta_1(1-\Delta) c(1-\Delta)\bar{Z} + \beta_4(1-\Delta) c(1-\Delta)^2 \bar{Z}^2 - c(1-\Delta)\bar{Z}\frac{a(1-\Delta) + c(1-\Delta)\bar{Z}}{b(1-\Delta)} \\
&+ \bigg(\beta_1(1-\Delta)(1 + c(1-\Delta)) - 2\beta_4(1-\Delta)\bar{Z}(1 + c(1-\Delta)) + c(1-\Delta)\frac{a(1-\Delta) + c(1-\Delta)\bar{Z}}{b(1-\Delta)}\bigg) z^i
\end{aligned}
$$

$$+\Bigg(\beta_1(1-\Delta)f(1-\Delta)+\beta_2(1-\Delta)-2\beta_4(1-\Delta)f(1-\Delta)c(1-\Delta)\bar{Z}-\beta_7(1-\Delta)\bar{Z}c(1-\Delta)$$
$$+f(1-\Delta)\frac{a(1-\Delta)+c(1-\Delta)\bar{Z}}{b(1-\Delta)}\Bigg)w^i$$
$$+\Bigg(-\beta_1(1-\Delta)f(1-\Delta)+\beta_3(1-\Delta)+2\beta_4(1-\Delta)f(1-\Delta)c(1-\Delta)\bar{Z}-\beta_8(1-\Delta)\bar{Z}c(1-\Delta)$$
$$-f(1-\Delta)\frac{a(1-\Delta)+c(1-\Delta)\bar{Z}}{b(1-\Delta)}-\frac{f(1-\Delta)c(1-\Delta)}{b(1-\Delta)}\bar{Z}\Bigg)\bar{W}$$
$$+\beta_4(1-\Delta)(1+c(1-\Delta))^2(z^j)^2$$
$$+\Big(\beta_4(1-\Delta)f(1-\Delta)^2+\beta_5(1-\Delta)+f(1-\Delta)\beta_7(1-\Delta)\Big)(w^j)^2$$
$$+\Bigg(\beta_4(1-\Delta)f(1-\Delta)^2-\beta_8(1-\Delta)f(1-\Delta)+\beta_6(1-\Delta)-\frac{f(1-\Delta)^2}{b(1-\Delta)}\Bigg)\bar{W}^2$$
$$+\Big(2\beta_4(1-\Delta)(1+c(1-\Delta))f(1-\Delta)+(1+c(1-\Delta))\beta_7(1-\Delta)\Big)z^jw^j$$
$$+\Bigg(-2\beta_4(1-\Delta)(1+c(1-\Delta))f(1-\Delta)+(1+c(1-\Delta))\beta_8(1-\Delta)+\frac{f(1-\Delta)c(1-\Delta)}{b(1-\Delta)}\Bigg)z^j\bar{W}$$
$$+\Bigg(-2\beta_4(1-\Delta)f(1-\Delta)^2-f(1-\Delta)\beta_7(1-\Delta)+f(1-\Delta)\beta_8(1-\Delta)+\beta_9(1-\Delta)+\frac{f(1-\Delta)^2}{b(1-\Delta)}\Bigg)w^j\bar{W}$$

Finally, accounting for the no-trade period, we have

$$e^{r\epsilon}J^d(t=1-\Delta-\epsilon,z^i,w^i,\bar{W})-(e^{r\epsilon}-1)\Bigg(z^i(v+w^i)-\frac{\gamma_d}{2r}\big(z^i\big)^2\Bigg)$$
$$=J^n(t=1-\Delta^-,z^i,w^i,\bar{W})+e^{r\epsilon}\lambda_d\sigma_d^2\epsilon\frac{r(N-2)}{2\gamma_d N}.$$

This equation gives the following boundary conditions at $t=1-\Delta$:

$$e^{r\epsilon}\alpha_0(1-\Delta-\epsilon)-e^{r\epsilon}\lambda_d\sigma_d^2\epsilon\frac{r(N-2)}{2\gamma_d N}=\beta_0(1-\Delta)$$
$$-\beta_1(1-\Delta)c(1-\Delta)\bar{Z}+\beta_4(1-\Delta)c(1-\Delta)^2\bar{Z}^2-c(1-\Delta)\bar{Z}\frac{a(1-\Delta)+c(1-\Delta)\bar{Z}}{b(1-\Delta)},$$
$$e^{r\epsilon}\alpha_1(1-\Delta-\epsilon)-(e^{r\epsilon}-1)v$$
$$=\beta_1(1-\Delta)(1+c(1-\Delta))-2\beta_4(1-\Delta)\bar{Z}(1+c(1-\Delta))+c(1-\Delta)\frac{a(1-\Delta)+c(1-\Delta)\bar{Z}}{b(1-\Delta)},$$
$$e^{r\epsilon}\alpha_2(1-\Delta-\epsilon)$$
$$=\beta_1(1-\Delta)f(1-\Delta)+\beta_2(1-\Delta)-2\beta_4(1-\Delta)f(1-\Delta)c(1-\Delta)\bar{Z}-\beta_7(1-\Delta)\bar{Z}c(1-\Delta)$$

$$+ f(1-\Delta)\frac{a(1-\Delta)+c(1-\Delta)\bar{Z}}{b(1-\Delta)},$$

$$e^{r\epsilon}\alpha_3(1-\Delta-\epsilon)$$

$$= -\beta_1(1-\Delta)f(1-\Delta) + \beta_3(1-\Delta) + 2\beta_4(1-\Delta)f(1-\Delta)c(1-\Delta)\bar{Z} - \beta_8(1-\Delta)\bar{Z}c(1-\Delta)$$

$$- f(1-\Delta)\frac{a(1-\Delta)+c(1-\Delta)\bar{Z}}{b(1-\Delta)} - \frac{f(1-\Delta)c(1-\Delta)}{b(1-\Delta)}\bar{Z},$$

$$e^{r\epsilon}\alpha_4(1-\Delta-\epsilon) + (e^{r\epsilon}-1)\frac{\gamma_d}{2r} = \beta_4(1-\Delta)(1+c(1-\Delta))^2,$$

$$e^{r\epsilon}\alpha_5(1-\Delta-\epsilon) = \beta_4(1-\Delta)f(1-\Delta)^2 + \beta_5(1-\Delta) + f(1-\Delta)\beta_7(1-\Delta),$$

$$e^{r\epsilon}\alpha_6(1-\Delta-\epsilon) = \beta_4(1-\Delta)f(1-\Delta)^2 - \beta_8(1-\Delta)f(1-\Delta) + \beta_6(1-\Delta) - \frac{f(1-\Delta)^2}{b(1-\Delta)},$$

$$e^{r\epsilon}\alpha_7(1-\Delta-\epsilon) - (e^{r\epsilon}-1) = 2\beta_4(1-\Delta)(1+c(1-\Delta))f(1-\Delta) + (1+c(1-\Delta))\beta_7(1-\Delta),$$

$$e^{r\epsilon}\alpha_8(1-\Delta-\epsilon)$$

$$= -2\beta_4(1-\Delta)(1+c(1-\Delta))f(1-\Delta) + (1+c(1-\Delta))\beta_8(1-\Delta) + \frac{f(1-\Delta)c(1-\Delta)}{b(1-\Delta)},$$

$$e^{r\epsilon}\alpha_9(1-\Delta-\epsilon) = -2\beta_4(1-\Delta)f(1-\Delta)^2$$

$$- f(1-\Delta)\beta_7(1-\Delta) + f(1-\Delta)\beta_8(1-\Delta) + \beta_9(1-\Delta) + \frac{f(1-\Delta)^2}{b(1-\Delta)}.$$

The boundary conditions at $t = 1$ are simply

$$\alpha_i(0) = \beta_i(1),$$

for $i = 0, 1 \ldots, 9$.

To summarize, we have specified 20 boundary conditions at times $1-\Delta$ and 1, along with 4 demand optimality conditions at time $1-\Delta$. There are four unknowns associated with the α_i's, 10 unknowns associated with the β_i's, 4 unknowns determining the demand functions at $1-\Delta$, and the length of the no-trade period ϵ. Thus, these unknowns are overdetermined. Let us specify how we solve the equations.

First, using the boundary conditions at time $1-\Delta$, for α_i for $i = 4, \ldots, 9$, one can solve for $B_4, \ldots, B_9$ in terms of ϵ. Imposing, for instance, $\beta_4(1) = \alpha_4$ yields a solution for ϵ. Then, one can verify that $\beta_i(1) = \alpha_i$ for $i = 5, \ldots, 9$. And, one can solve the four demand optimality conditions for a, b, c, f at time $1-\Delta$.

Now, there are 8 remaining unknowns, A_i, B_i, for $i = 0, 1, 2, 3$, which determine α_i, β_i, for $i = 0, 1, 2, 3$. These unknowns solve four boundary conditions at $1-\Delta$ and four boundary conditions at time 1. This completes the solution of the model.

We conclude by providing several expressions for some of the quantities in the model.

$$c(1-\Delta) = -\frac{(N-2)(1-e^{-\Delta r})}{e^{-\Delta r} + (1-e^{-\Delta r})(N-1)},$$

$$\epsilon = \max\Bigg\{0, \min\Bigg\{1-\Delta, \frac{1}{r}\log\left[\frac{(N-1)\big(\gamma_d - \gamma_n(1+c(1-\Delta))^2\big) - e^{-r\Delta}(1+c(1-\Delta))^2(\gamma_d - \gamma_n(N-1))}{\gamma_d(N-2)}\right]\Bigg\}\Bigg\}.$$

Assume that $\bar{Z} = 0$ and $v = 0$, then A_0 is simply

$$A_0 = \frac{(N-2)\big(e^r \lambda_d \sigma_d^2 + e^{r(\Delta+\epsilon)}\lambda_d \sigma_d^2(\epsilon r - 1) + \Delta r \lambda_n \sigma_n^2\big)}{2\gamma_d(e^r - 1)N}. \tag{17}$$

The value function during the no-trade period itself is

$$J^d(t, z^i, w^i, \bar{W}) = (1 - e^{-r(1-\Delta-t)})\left(r(v + w^i)z^i - \frac{\gamma_d}{2r}\big(z^i\big)^2\right) + e^{-r(1-\Delta-t)}J^n(t = 1-\Delta^-, z^i, w^i, \bar{W}) + e^{-r(1-\Delta-t)}\lambda_d \sigma_d^2(1-\Delta-t)e^{r\epsilon}\frac{r(N-2)}{2\gamma_d N}.$$

The average welfare during trade is

$$W(\Delta) := \frac{1}{1-\Delta}\int_0^{1-\Delta} \mathbb{E}\left[J^d(t, 0, w^i, \bar{W})\right]dt.$$

IA.4.1 Volume

Assume $\lambda_d = \bar{\lambda}_d \ell$, and $\sigma_d^2 = \frac{1}{\ell}\bar{\sigma}_d^2$ for some $\ell, \bar{\lambda}_d, \bar{\sigma}_d$. Then, letting $\ell \to \infty$, w^j and $\bar{W}^j$ converge in law to Brownian motions during the day. We'll restrict attention to this limiting case both during the day and night, as it makes the computation of expressions involving volume much more tractable. Additionally, we'll assume $\gamma_n = \gamma_d$, which is sufficient to ensure volume reaches a steady state distribution. Denote the volatility of the Brownian shocks during the day and night by σ_d, σ_n, respectively.

We will omit time subscripts when denoting demand coefficients in the portion of the day preceding the no-trade period, since those coefficients are constant, and denote coefficients at the closing session by a $1-\Delta$ subscript. Then, during the trading day,

$$D_t^i = D_0^i e^{ct} + f\sigma_d \int_0^t e^{c(t-s)}(dw_s^i - d\bar{W}_s).$$

In addition, under the assumption that $\gamma_d = \gamma_n$, we have $f_{1-\Delta}/c_{1-\Delta} = f/c$. Using this, one can show

$$D^i_{1-\Delta} = \frac{c_{1-\Delta}}{c} D^i_{1-\Delta-\epsilon} + f_{1-\Delta}\sqrt{\epsilon}\sigma_d \sqrt{\frac{N-1}{N}}\delta_2$$

for some $N(0,1)$ variable δ_2. Moreover,

$$D^i_1 = (1 + c_{1-\Delta})D^i_{1-\Delta-\epsilon} + (cf_{1-\Delta} + f)\sqrt{\epsilon}\sigma_d \sqrt{\frac{N-1}{N}}\delta_2 + f\sqrt{\Delta}\sigma_n \sqrt{\frac{N-1}{N}}\delta_3,$$

for an independent $N(0,1)$ shock δ_3. Combining these expressions,

$$\begin{aligned} D^i_1 = (1 + c_{1-\Delta})D^i_0 e^{c(1-\Delta-\epsilon)} &+ (1 + c_{1-\Delta})f\sigma_d \sqrt{\frac{N-1}{N}} \sqrt{\frac{e^{2c(1-\Delta-\epsilon)} - 1}{2c}}\delta_1 \\ &+ (cf_{1-\Delta} + f)\sqrt{\epsilon}\sigma_d \sqrt{\frac{N-1}{N}}\delta_2 + f\sqrt{\Delta}\sigma_n \sqrt{\frac{N-1}{N}}\delta_3, \end{aligned}$$

for a third independent $N(0,1)$ shock δ_1.

Therefore, $(D^i_n)^\infty_{n=0}$ is an AR(1) process with normally distributed shocks. Moreover, values of D^i_t throughout the day have an unconditional normal distribution with mean 0 and variance

$$e^{2ct}\frac{((1 + c_{1-\Delta})^2 f^2\sigma_d^2 \frac{e^{2c(1-\Delta-\epsilon)}-1}{2c} + (cf_{1-\Delta} + f)^2\epsilon\sigma_d^2 + f^2\Delta\sigma_n^2)\frac{N-1}{N}}{1 - (1 + c_{1-\Delta})^2 e^{2c(1-\Delta-\epsilon)}} + f^2\sigma_d^2 \frac{N-1}{N}\frac{e^{2ct} - 1}{2c}.$$

Thus, since volume at any point of the day is simply $\sum_i |D^i_t|$, its expectation is the mean of (the sum of) a folded normal distribution.

IA.4.2 The Exchange's Problem

In this subsection, we formally model an approximation of an exchange's problem. Assume that an exchange's goal is to maximize the expected volume. They do this by choosing the length of closure, Δ. We abuse notation below, but note that $D^i_t = 0$ during any period of no trade or during the closure. The exchange's problem is

$$\Delta^E \in \underset{\Delta\in[0,1)}{\operatorname{argmax}} \frac{1}{1-\Delta}\int_0^{1-\Delta} E\left[\int_t^\infty e^{-r(s-t)}\left(\sum_{i=1}^N \left|D^i_s\right|\right) ds\right] dt. \tag{18}$$

The inner internal is the realized volume over the life of the asset. The outer integral averages over the start time in the first day, as time is a state variable. Note the integral above abuses notation, since trades at the end-of-day session are discrete quantities, not flows. These discrete trades can be thought of as Dirac delta functions in the integral above.

We can rewrite the problem as

$$\Delta^E \in \underset{\Delta\in[0,1)}{\operatorname{argmax}} \frac{1}{1-\Delta}\int_0^{1-\Delta}\int_t^\infty e^{-r(s-t)}\left(\sum_{i=1}^N E\left|D_s^i\right|\right) ds\ dt.$$

As traders are ex-ante identical, $E\left[|D_t^i|\right]$ is unconditionally symmetric across traders. Also, from IA.4.1, we know that D_t^i is unconditionally normal with mean 0 and a time-dependent variance, denoted σ_t^2. Therefore, $|D_t^i|$ is a folded normal distribution with mean $\sqrt{\frac{2}{\pi}}\sigma_t$. The problem reduces to

$$\Delta^E \in \underset{\Delta\in[0,1)}{\operatorname{argmax}} N\sqrt{\frac{2}{\pi}}\frac{1}{1-\Delta}\int_0^{1-\Delta}\int_t^\infty e^{-r(s-t)}\sigma_s ds\ dt.$$

Finally, let's take care of the discrete trades at the close of each day. Breaking the inner integral into pieces, it becomes

$$\int_t^{1-\Delta} e^{-r(s-t)}\sigma_s ds + e^{-r(1-\Delta-t)}\sigma_{1-\Delta} + \sum_{k=1}^\infty\left[\int_k^{k+1-\Delta-\epsilon} e^{-r(s-t)}\sigma_s ds + e^{-r(k+1-\Delta)}\sigma_{k+1-\Delta}\right]$$

where for $k \geq 0$ and for $t \in [k, k+1-\Delta-\epsilon]$, σ_t is

$$\sqrt{e^{2c(t-k)}\frac{((1+c_{1-\Delta})^2 f^2\sigma_d^2\frac{e^{2c(1-\Delta-\epsilon)}-1}{2c} + (cf_{1-\Delta}+f)^2\epsilon\sigma_d^2 + f^2\Delta\sigma_n^2)\frac{N-1}{N}}{1-(1+c_{1-\Delta})^2 e^{2c(1-\Delta-\epsilon)}} + f^2\sigma_d^2\frac{N-1}{N}\frac{e^{2c(t-k)}-1}{2c}},$$

for $t \in (k+1-\Delta-\epsilon, k+1-\Delta)$, $\sigma_t = 0$, and for $t = k+1-\Delta$, σ_t is

$$\sqrt{\left(\frac{c_{1-\Delta}}{c}\right)^2\sigma_{k+1-\Delta-\epsilon}^2 + \frac{f_{1-\Delta}^2\epsilon(N-1)}{N}\sigma_d^2}.$$

Note that γ just scales volume and only the ratio of volatility between day and night matters, not the levels. Therefore, the optimal length of closure from an exchange's perspective, Δ^E, is only a function of N, r, and $\frac{\sigma_d}{\sigma_n}$.

This extension allows us to compare trader-optimal, Δ^*, and exchange-optimal, Δ^E, closures numerically. Figure D.1 plots these two quantities as a function of the number of traders on the exchange, N. In general, there does not seem to be an interior optimum for an exchange. When the market is small, they maximize volume by having one discrete trading session at the start of the day. When the market is sufficiently large, volume is then maximized by having trade 24/7. In practice, most exchanges require some downtime for basic daily maintenance, which they prefer not to have during trading in case of technical issues.[14] In general, the decision of the optimal length of closure is positively related between

[14]For example, the CME Globex Trading System closes from 5:00 to 6:00 p.m. EST for daily maintenance.

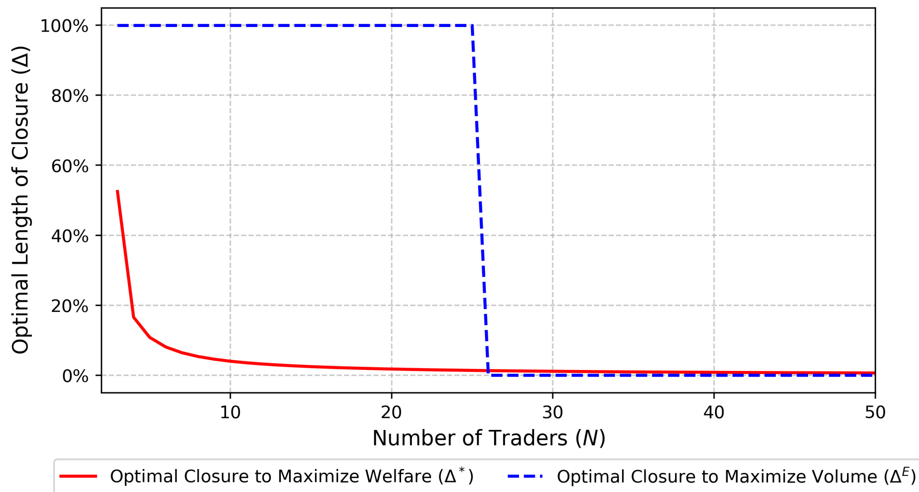

Figure D.1. Optimal Length of Closure
We plot the ex-ante welfare maximizing length of closure, Δ^*, that maximizes Equation 16 and the optimal length of closure, Δ^E, is that which maximizes expected volume defined by Equation 18. We assume that σ and γ are constant across day and night and use $r = 10\%$.

traders and an exchange.

Finally, we use the calibrated quantities from Table I to see what the exchanges would prefer to do within the confines of our model. The results are in Table D.1. Given that all four exchanges are calibrated to be large, it is not surprising that all four calibrations imply that 24/7 trading is optimal from an exchange's perspective. In fact, the implied increase in volume from extending trading hours to 23/7 or 24/7 is very large, ranging from 74.9% up to 90.5%. It is, therefore, not surprising that three of these four exchanges already have plans to extend their trading hours. However, a naive estimate for the increase in volume from extending hours would be on the order of 254% as that is the increase in the amount of trading hours a day from 6.5 to 23. We only calibrate a third of that effect due to the endogeneity in the trading strategies. As hours are extended, per-period liquidity drops, especially at the close, making instantaneous volume much smaller.

IA.4.3 Convergence

In this section, we show numerically that the discrete trade model converges to the

Table D.1
Calibration: Exchange's Perspective

This table compares the volume of the current market closure to that of 23/7 trading, or the optimal length of closure from an exchange's perspective by using the calibrated volatility and number of traders per exchange. $\hat{N}$ denotes the estimated size of the market, and $\frac{\hat{\sigma_d}}{\sigma_n}$ is the relative instantaneous volatilities during the day and night. We assume that total volatility is constant across closure lengths so that σ_d solves $\sigma_T^2 = (1-\Delta)\sigma_d^2 + \Delta\sigma_n^2$. The optimal length of closure, Δ^E, is that which maximizes expected volume defined by Equation 18 given the calibrated parameters and subject to the total volatility constraint. We assume $r = 10\%$, $v = 0$, and $z_0^i = 0$ for all calibrations.

Exchange	**Current Length of Night (Δ)**	$\widehat{N}$	$\widehat{\frac{\sigma_d}{\sigma_n}}$	**Optimal Length of Night (Δ^E)**	**% Volume Change from Δ to 23/7**	**% Volume Change from Δ to Δ^E**
NYSE	72.9%	208	1.28	0.0%	74.9%	75.5%
Nasdaq	72.9%	325	1.32	0.0%	89.1%	90.1%
Arca	72.9%	303	1.23	0.0%	89.6%	90.5%
CBOE EDGX	72.9%	191	0.87	0.0%	83.6%	84.1%

continuous trade model. In particular, for a given set of parameter values, Figure D.2 plots the maximum difference between the discrete and continuous trade welfares, end-of-day aggressiveness captured by c in the final session, and no-trade period lengths. This maximum is over $\Delta \in \{0, 1, \ldots, K-1\}$, where for the continuous trade model, Δ is replaced by Δ/K. As we see, the errors follow roughly a linear path in the log-log plots, suggesting convergence is algebraic.

IA.4.4 Proof of Proposition 4: Existence of Non-Zero Optimum

In this section, we prove Proposition 3. Specifically, we show that a very short closure always increases welfare relative to 24/7 trading, and therefore, the optimal length of closure is never zero. First, we take the derivative of welfare, Equation 16, with respect to the closure length,

$$\frac{\partial}{\partial \Delta} W(\Delta) = \frac{\partial}{\partial \Delta}\left[\frac{1}{1-\Delta}\int_0^{1-\Delta} N\alpha_0(t) + \sigma^2\Big(N\alpha_5(t) + \alpha_6(t) + \alpha_9(t)\Big)dt\right],$$

where the α's are defined in Appendix IA.4. For algebraic simplicity, we won't write out $\epsilon(\Delta)$, but note that $\epsilon(\Delta = 0) = 0$. We will also be focusing on cases with very small Δ, and

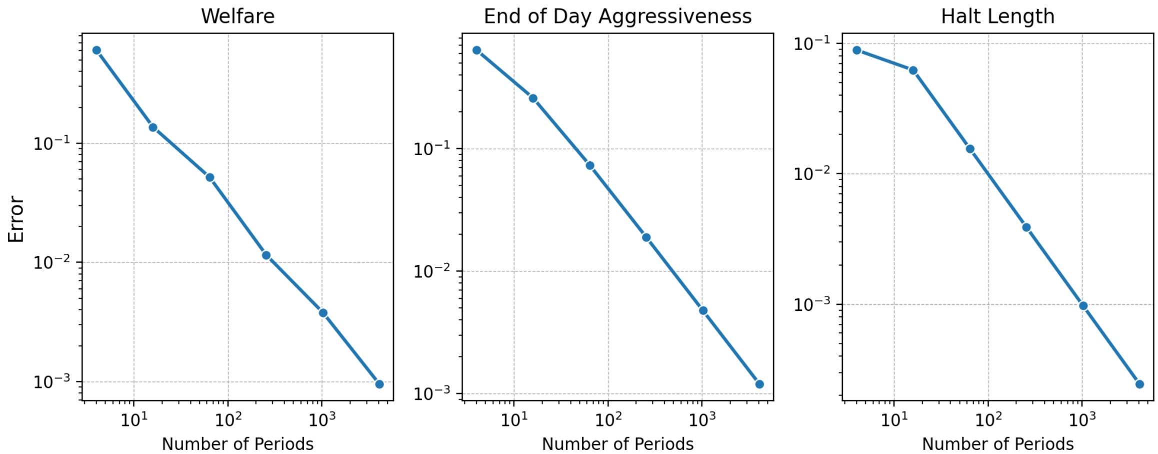

Figure D.2. Convergence of Discrete Trade Solution

This figure plots maximum absolute errors in various characteristics of the discrete and continuous trade models as a function of the number of trading periods in a day. The maximum is over the length of the trading day, and errors are given as a function of K, the number of periods in the trading day. K is set to 4^i, for $i = 1, \ldots, 6$. We set $r = 10\%, \lambda = 1, \sigma = 1, \gamma = 1, N = 100$.

so ϵ will always be less than $1 - \Delta$. After some simplifications, the derivative of welfare can be written as

$$\begin{aligned}\frac{\partial}{\partial \Delta} W(\Delta) =& \frac{e^{-r\Delta}(N-2)\sigma^2}{2(1-\Delta)^2(e^r-1)\gamma r}\Bigg(e^{r(1+\Delta+\epsilon(\Delta))}(\lambda+r) - e^{r+\Delta r}(2\lambda+r) \\ & - e^{r(\Delta+\epsilon(\Delta))}(2\lambda+r) + e^{r(2\Delta+\epsilon(\Delta))}\lambda(1+r-\Delta r) + e^{\Delta r}(\lambda+r-\lambda r) \\ & + e^r\lambda\left(1+\Delta r\left(1-(1-\Delta)r\right)\right) + e^{\Delta r} r\Bigg[(1-\Delta)(e^r-1)(e^{r\epsilon(\Delta)}-1)(\lambda+r)\,\epsilon'(\Delta) \\ & + \epsilon(\Delta)\Bigg(r - e^r r + \lambda\left(1 - e^r + e^{r\epsilon(\Delta)} + e^{r(\Delta+\epsilon(\Delta))}(-1+(-1+\Delta)r)\right) \\ & + (-1+\Delta)e^{r\epsilon(\Delta)}(-1+e^{\Delta r})\lambda r\,\epsilon'(\Delta)\Bigg)\Bigg]\Bigg).\end{aligned}$$

First, note that

$$\frac{\partial}{\partial \Delta} W(\Delta)\Big|_{\Delta=0} = 0.$$

Then, taking another derivative to get the second-order condition and evaluating at $\Delta = 0$, we get that

$$\frac{\partial^2}{\partial \Delta^2} W(\Delta)\bigg|_{\Delta=0} = \frac{(N-2)r^2\sigma^2}{2\gamma} > 0.$$

Therefore, welfare is strictly convex at $\Delta = 0$. So, $W(\Delta') > W(0)$ for some Δ' sufficiently small, and, therefore, the optimal length of closure is non-zero.

IA.5 Simplifications of Discrete Trade Solutions

IA.5.1 Simplifications of Model without Information:

Let's simplify some of the recursions describing the value function by using the FOCs:

$$a_0^k = \bar{Z}c_k\bigg(-\frac{a_k + c_k\bar{Z}}{b_k} - (1-e^{-rh})v - \frac{(1-e^{-rh})\gamma_d}{2r}c_k\bar{Z} - e^{-rh}a_1^{k+1} + e^{-rh}a_4^{k+1}c_k\bar{Z}\bigg)$$

$$+ e^{-rh}a_0^{t+1} + e^{-rh}a_5^{k+1}\sigma^2 + e^{-rh}a_6^{k+1}\frac{\sigma^2}{N} + e^{-rh}a_9^{k+1}\frac{\sigma^2}{N}$$

$$= -\bar{Z}^2c_k^2\bigg(\frac{1}{b_k(N-1)} + e^{-rh}a_4^{k+1}\bigg) + e^{-rh}a_0^{t+1} + e^{-rh}a_5^{k+1}\sigma^2 + e^{-rh}a_6^{k+1}\frac{\sigma^2}{N} + e^{-rh}a_9^{k+1}\frac{\sigma^2}{N}$$

and

$$\begin{aligned}
a_1^k &= \frac{c_k a_k + c_k^2\bar{Z}}{b_k} + (1-e^{-rh})(1+c_k)v \\
&\quad + \frac{(1-e^{-rh})\gamma_d}{r}(1+c_k)c_k\bar{Z} + e^{-rh}(1+c_k)a_1^{k+1} - 2e^{-rh}(1+c_k)c_k\bar{Z}a_4^{k+1} \\
&= c_k\left(\frac{c_k\bar{Z}}{b_k(N-1)} - (1-e^{-rh})v - \frac{(1-e^{-rh})\gamma_d c_k\bar{Z}}{r} - e^{-rh}a_1^{k+1} + 2e^{-rh}a_4^{k+1}c_k\bar{Z}\right) \\
&\quad + (1-e^{-rh})(1+c_k)v + \frac{(1-e^{-rh})\gamma_d}{r}(1+c_k)c_k\bar{Z} + e^{-rh}(1+c_k)a_1^{k+1} - 2e^{-rh}(1+c_k)c_k\bar{Z}a_4^{k+1} \\
&= \frac{c_k^2\bar{Z}}{b_k(N-1)} + (1-e^{-rh})v + \frac{(1-e^{-rh})\gamma_d}{r}c_k\bar{Z} + e^{-rh}a_1^{k+1} - 2e^{-rh}c_k\bar{Z}a_4^{k+1} \\
&= \frac{c_k(c_k+1)\bar{Z}}{b_k(N-1)} - \frac{a_k + c_k\bar{Z}}{b_k}
\end{aligned}$$

and

$$a_2^k = \frac{f_k a_k}{b_k} + \frac{f_k c_k}{b_k}\bar{Z} + (1-e^{-rh})(f_k v - c_k\bar{Z}) + \frac{(1-e^{-rh})\gamma_d}{r}c_k f_k\bar{Z} + e^{-rh}f_k a_1^{k+1} + e^{-rh}a_2^{k+1}$$

$$
\begin{aligned}
&- e^{-rh}2a_4^{k+1}c_kf_k\bar{Z} - e^{-rh}a_7^{k+1}c_k\bar{Z}\\
&= f_k\left(\frac{a_k}{b_k} + \frac{c_k}{b_k}\bar{Z} + (1-e^{-rh})v + \frac{(1-e^{-rh})\gamma_d}{r}c_k\bar{Z} + e^{-rh}a_1^{k+1} - e^{-rh}2a_4^{k+1}c_k\bar{Z}\right)\\
&- (1-e^{-rh})c_k\bar{Z} + e^{-rh}a_2^{k+1} - e^{-rh}a_7^{k+1}c_k\bar{Z}\\
&= f_k\frac{c_k\bar{Z}}{b_k(N-1)} - (1-e^{-rh})c_k\bar{Z} + e^{-rh}a_2^{k+1} - e^{-rh}a_7^{k+1}c_k\bar{Z}
\end{aligned}
$$

$$
\begin{aligned}
a_3^k &= -\frac{f_ka_k}{b_k} - 2\frac{f_kc_k}{b_k}\bar{Z} - (1-e^{-rh})f_kv - \frac{(1-e^{-rh})\gamma_d}{r}c_kf_k\bar{Z} - e^{-rh}f_ka_1^{k+1} + e^{-rh}a_3^{k+1}\\
&+ e^{-rh}2a_4^{k+1}c_kf_k\bar{Z} - e^{-rh}a_8^{k+1}c_k\bar{Z}\\
&= -f_k\left(\frac{a_k}{b_k} + \frac{c_k\bar{Z}}{b_k} + (1-e^{-rh})v + \frac{(1-e^{-rh})\gamma_d}{r}c_k\bar{Z} + e^{-rh}a_1^{k+1} - e^{-rh}2a_4^{k+1}c_k\bar{Z}\right)\\
&- \frac{f_kc_k}{b_k}\bar{Z} + e^{-rh}a_3^{k+1} - e^{-rh}a_8^{k+1}c_k\bar{Z}\\
&= -\frac{c_kf_k\bar{Z}}{b_k(N-1)} - \frac{f_kc_k}{b_k}\bar{Z} + e^{-rh}a_3^{k+1} - e^{-rh}a_8^{k+1}c_k\bar{Z}
\end{aligned}
$$

and adding these last two, and using the solution to f/b by adding the last two optimality of demand, $a_2^k + a_3^k = e^{-rh}(a_2^{k+1} + a_3^{k+1})$, which implies $a_2 = -a_3$.

$$
\begin{aligned}
a_5^t &= (1-e^{-rh})f_k - \frac{(1-e^{-rh})\gamma_d}{2r}f_k^2 + e^{-rh}a_4^{k+1}f_k^2 + e^{-rh}a_5^{k+1} + e^{-rh}a_7^{k+1}f_k\\
&= (1-e^{-rh})\frac{f_k}{2} - \frac{f_k^2}{2b_k(N-1)} + e^{-rh}a_7^{k+1}\frac{f_k}{2} + e^{-rh}a_5^{k+1}
\end{aligned}
$$

$$
\begin{aligned}
a_6^k &= -\frac{f_k^2}{b_k} - \frac{(1-e^{-rh})\gamma_d}{2r}f_k^2 + e^{-rh}a_4^{k+1}f_k^2 + e^{-rh}a_6^{k+1} - e^{-rh}a_8^{k+1}f_k\\
&= -\frac{f_k^2N}{2b_k(N-1)} - e^{-rh}a_8^{k+1}\frac{f_k}{2} + e^{-rh}a_6^{k+1}
\end{aligned}
$$

These two imply $a_5^k - a_6^k = -\frac{f_k}{2} + e^{-rh}(a_5^{k+1} - a_6^{k+1})$.

$$
\begin{aligned}
a_7^k &= (1-e^{-rh})(1+c_k) - \frac{(1-e^{-rh})\gamma_d}{r}(1+c_k)f_k + 2e^{-rh}a_4^{k+1}(1+c_k)f_k + e^{-rh}a_7^{k+1}(1+c_k)\\
&= -\frac{f_k}{b_k(N-1)}(1+c_k)
\end{aligned}
$$

and

$$
a_8^k = \frac{f_k(1+c_k)}{b_k(N-1)} - \frac{f_k}{b_k}
$$

Adding the equations for a_7, a_8,

$$a_7^k + a_8^k = -\frac{f_k}{b_k} = (1 - e^{-rh}) + e^{-rh}(a_7^{k+1} + a_8^{k+1}).$$

This implies $a_7 + a_8 = 1$. Last,

$$\begin{aligned}
a_9^k = \frac{f_k^2}{b_k} - (1-e^{-rh})f_k + \frac{(1-e^{-rh})\gamma_d}{r} f_k^2 - 2e^{-rh}a_4^{k+1}f_k^2 - e^{-rh}a_7^{k+1}f_k + e^{-rh}a_8^{k+1}f_k + e^{-rh}a_9^{k+1} \\
= \frac{f_k^2}{b_k} + \frac{f_k^2}{b_k(N-1)} + e^{-rh}a_8^{k+1}f_k + e^{-rh}a_9^{k+1} \\
= \frac{2f_k^2}{b_k(N-1)} - \frac{(1-e^{-rh})\gamma_d f_k^2}{r} + 2e^{-rh}a_4^{k+1}f_k^2 + e^{-rh}a_9^{k+1} \\
= \frac{f_k^2(2+c_k)}{b_k(1+c_k)(N-1)} + e^{-rh}a_9^{k+1}
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
a_0^k &= -\bar{Z}^2 c_k^2 \left(\frac{1}{b_k(N-1)} + e^{-rh}a_4^{k+1} \right) + e^{-rh}a_0^{t+1} + e^{-rh}a_5^{k+1}\sigma^2 + e^{-rh}a_6^{k+1}\frac{\sigma^2}{N} + e^{-rh}a_9^{k+1}\frac{\sigma^2}{N} \\
a_1^k &= \frac{c_k(c_k+1)\bar{Z}}{b_k(N-1)} - \frac{a_k + c_k\bar{Z}}{b_k} \\
a_2^k &= f_k \frac{c_k\bar{Z}}{b_k(N-1)} - (1-e^{-rh})c_k\bar{Z} + e^{-rh}a_2^{k+1} - e^{-rh}a_7^{k+1}c_k\bar{Z} \\
a_3^k &= -\frac{c_k f_k N\bar{Z}}{b_k(N-1)} + e^{-rh}a_3^{k+1} - e^{-rh}a_8^{k+1}c_k\bar{Z} \\
a_4^k &= -\frac{(1-e^{-rh})\gamma_d}{2r}(1+c_k)^2 + e^{-rh}a_4^{k+1}(1+c_k)^2 \\
a_5^k &= (1-e^{-rh})\frac{f_k}{2} - \frac{f_k^2}{2b_k(N-1)} + e^{-rh}a_7^{k+1}\frac{f_k}{2} + e^{-rh}a_5^{k+1} \\
a_6^k &= -\frac{f_k^2 N}{2b_k(N-1)} - e^{-rh}a_8^{k+1}\frac{f_k}{2} + e^{-rh}a_6^{k+1} \\
a_7^k &= -\frac{f_k}{b_k(N-1)}(1+c_k) \\
a_8^k &= \frac{c_k f_k}{b_k} - \frac{f_k(N-2)}{b_k(N-1)}(1+c_k) \\
a_9^k &= \frac{f_k^2(2+c_k)}{b_k(1+c_k)(N-1)} + e^{-rh}a_9^{k+1}
\end{aligned}$$

for $t < T$. There is an analogous recursion at time T.

IA.5.2 Simplifications of Model with Information:

The recursions corresponding to the Bellman equation are given by

$$a_0^k = -\bar{Z}\frac{c_k a_k + c_k^2\bar{Z}}{b_k} - \frac{(1-e^{-rh})\gamma_d}{2r}c_k^2\bar{Z}^2$$
$$+ e^{-rh}a_0^{t+1} - e^{-rh}a_1^{k+1}c_k\bar{Z} + e^{-rh}a_4^{k+1}c_k^2\bar{Z}^2 + e^{-rh}a_5^{k+1}\lambda\sigma^2 + e^{-rh}a_6^{k+1}\lambda\sigma_N^2 + e^{-rh}a_9^{k+1}\lambda\sigma_N^2$$
$$a_1^k = \frac{c_k a_k + c_k^2\bar{Z}}{b_k} + \frac{(1-e^{-rh})\gamma_d}{r}(1+c_k)c_k\bar{Z} + e^{-rh}(1+c_k)a_1^{k+1} - 2e^{-rh}(1+c_k)c_k\bar{Z}a_4^{k+1}$$
$$a_2^k = \frac{f_k a_k}{b_k} + \frac{f_k c_k}{b_k}\bar{Z} - (1-e^{-rh})c_k\bar{Z}\frac{N\alpha-1}{N-1} + \frac{(1-e^{-rh})\gamma_d}{r}c_k f_k\bar{Z} + e^{-rh}f_k a_1^{k+1} + e^{-rh}a_2^{k+1}$$
$$- e^{-rh}2a_4^{k+1}c_k f_k\bar{Z} - e^{-rh}a_7^{k+1}c_k\bar{Z}$$
$$a_3^k = -\frac{f_k a_k}{b_k} - 2\frac{f_k c_k}{b_k}\bar{Z} - (1-e^{-rh})c_k\bar{Z}\frac{N(1-\alpha)}{N-1} - \frac{(1-e^{-rh})\gamma_d}{r}c_k f_k\bar{Z} - e^{-rh}f_k a_1^{k+1}$$
$$+ e^{-rh}a_3^{k+1} + e^{-rh}2a_4^{k+1}c_k f_k\bar{Z} - e^{-rh}a_8^{k+1}c_k\bar{Z}$$
$$a_4^k = -\frac{(1-e^{-rh})\gamma_d}{2r}(1+c_k)^2 + e^{-rh}a_4^{k+1}(1+c_k)^2$$
$$a_5^k = (1-e^{-rh})f_k\frac{N\alpha-1}{N-1} - \frac{(1-e^{-rh})\gamma_d}{2r}f_k^2 + e^{-rh}a_4^{k+1}f_k^2 + e^{-rh}a_5^{k+1} + e^{-rh}a_7^{k+1}f_k$$
$$a_6^k = -\frac{f_k^2}{b_k} - (1-e^{-rh})f_k\frac{N(1-\alpha)}{N-1} - \frac{(1-e^{-rh})\gamma_d}{2r}f_k^2 + e^{-rh}a_4^{k+1}f_k^2 + e^{-rh}a_6^{k+1} - e^{-rh}a_8^{k+1}f_k$$
$$a_7^k = (1-e^{-rh})(1+c_k)\frac{N\alpha-1}{N-1} - \frac{(1-e^{-rh})\gamma_d}{r}(1+c_k)f_k + 2e^{-rh}a_4^{k+1}(1+c_k)f_k$$
$$+ e^{-rh}a_7^{k+1}(1+c_k)$$
$$a_8^k = \frac{c_k f_k}{b_k} + (1-e^{-rh})(1+c_k)\frac{N(1-\alpha)}{N-1} + \frac{(1-e^{-rh})\gamma_d}{r}(1+c_k)f_k$$
$$- 2e^{-rh}a_4^{k+1}(1+c_k)f_k + e^{-rh}a_8^{k+1}(1+c_k)$$
$$a_9^k = \frac{f_k^2}{b_k} - (1-e^{-rh})f_k\frac{N\alpha-1}{N-1} + (1-e^{-rh})f_k\frac{N(1-\alpha)}{N-1} + \frac{(1-e^{-rh})\gamma_d}{r}f_k^2 - 2e^{-rh}a_4^{k+1}f_k^2$$
$$- e^{-rh}a_7^{k+1}f_k + e^{-rh}a_8^{k+1}f_k + e^{-rh}a_9^{k+1}.$$

Let's again simplify some of these recursions by using the FOCs:

$$a_0^k = \bar{Z}c_k\left(- \frac{a_k + c_k\bar{Z}}{b_k} - \frac{(1-e^{-rh})\gamma_d}{2r}c_k\bar{Z} - e^{-rh}a_1^{k+1} + e^{-rh}a_4^{k+1}c_k\bar{Z}\right)$$

$$
\begin{aligned}
&\quad + e^{-rh}a_0^{t+1} + e^{-rh}a_5^{k+1}\sigma^2 + e^{-rh}a_6^{k+1}\sigma_N^2 + e^{-rh}a_9^{k+1}\sigma_N^2 \\
&= -\bar{Z}^2 c_k^2\left(\frac{1}{b_k(N-1)} + e^{-rh}a_4^{k+1}\right) + e^{-rh}a_0^{t+1} + e^{-rh}a_5^{k+1}\sigma^2 + e^{-rh}a_6^{k+1}\sigma_N^2 + e^{-rh}a_9^{k+1}\sigma_N^2
\end{aligned}
$$

and

$$
\begin{aligned}
a_1^k &= \frac{c_k a_k + c_k^2\bar{Z}}{b_k} \\
&\quad + \frac{(1-e^{-rh})\gamma_d}{r}(1+c_k)c_k\bar{Z} + e^{-rh}(1+c_k)a_1^{k+1} - 2e^{-rh}(1+c_k)c_k\bar{Z}a_4^{k+1} \\
&= c_k\left(\frac{c_k\bar{Z}}{b_k(N-1)} - \frac{(1-e^{-rh})\gamma_d c_k\bar{Z}}{r} - e^{-rh}a_1^{k+1} + 2e^{-rh}a_4^{k+1}c_k\bar{Z}\right) \\
&\quad + \frac{(1-e^{-rh})\gamma_d}{r}(1+c_k)c_k\bar{Z} + e^{-rh}(1+c_k)a_1^{k+1} - 2e^{-rh}(1+c_k)c_k\bar{Z}a_4^{k+1} \\
&= \frac{c_k^2\bar{Z}}{b_k(N-1)} + \frac{(1-e^{-rh})\gamma_d}{r}c_k\bar{Z} + e^{-rh}a_1^{k+1} - 2e^{-rh}c_k\bar{Z}a_4^{k+1} \\
&= \frac{c_k(c_k+1)\bar{Z}}{b_k(N-1)} - \frac{a_k + c_k\bar{Z}}{b_k}
\end{aligned}
$$

and

$$
\begin{aligned}
a_2^k &= \frac{f_k a_k}{b_k} + \frac{f_k c_k}{b_k}\bar{Z} - (1-e^{-rh})c_k\bar{Z}\frac{N\alpha-1}{N-1} \\
&\quad + \frac{(1-e^{-rh})\gamma_d}{r}c_k f_k\bar{Z} + e^{-rh}f_k a_1^{k+1} + e^{-rh}a_2^{k+1} - e^{-rh}2a_4^{k+1}c_k f_k\bar{Z} - e^{-rh}a_7^{k+1}c_k\bar{Z} \\
&= f_k\left(\frac{a_k}{b_k} + \frac{c_k}{b_k}\bar{Z} + \frac{(1-e^{-rh})\gamma_d}{r}c_k\bar{Z} + e^{-rh}a_1^{k+1} - e^{-rh}2a_4^{k+1}c_k\bar{Z}\right) \\
&\quad - (1-e^{-rh})c_k\bar{Z}\frac{N\alpha-1}{N-1} + e^{-rh}a_2^{k+1} - e^{-rh}a_7^{k+1}c_k\bar{Z} \\
&= f_k\frac{c_k\bar{Z}}{b_k(N-1)} - (1-e^{-rh})c_k\bar{Z}\frac{N\alpha-1}{N-1} + e^{-rh}a_2^{k+1} - e^{-rh}a_7^{k+1}c_k\bar{Z}
\end{aligned}
$$

$$
\begin{aligned}
a_3^k &= -\frac{f_k a_k}{b_k} - 2\frac{f_k c_k}{b_k}\bar{Z} - (1-e^{-rh})c_k\bar{Z}\frac{N(1-\alpha)}{N-1} - \frac{(1-e^{-rh})\gamma_d}{r}c_k f_k\bar{Z} \\
&\quad - e^{-rh}f_k a_1^{k+1} + e^{-rh}a_3^{k+1} + e^{-rh}2a_4^{k+1}c_k f_k\bar{Z} - e^{-rh}a_8^{k+1}c_k\bar{Z} \\
&= -f_k\left(\frac{a_k}{b_k} + \frac{c_k\bar{Z}}{b_k} + \frac{(1-e^{-rh})\gamma_d}{r}c_k\bar{Z} + e^{-rh}a_1^{k+1} - e^{-rh}2a_4^{k+1}c_k\bar{Z}\right) \\
&\quad - \frac{f_k c_k}{b_k}\bar{Z} - (1-e^{-rh})c_k\bar{Z}\frac{1-\alpha}{N-1} + e^{-rh}a_3^{k+1} - e^{-rh}a_8^{k+1}c_k\bar{Z}
\end{aligned}
$$

$$= -\frac{c_k f_k \bar{Z}}{b_k(N-1)} - \frac{f_k c_k}{b_k}\bar{Z} - (1-e^{-rh})c_k\bar{Z}\frac{N(1-\alpha)}{N-1} + e^{-rh}a_3^{k+1} - e^{-rh}a_8^{k+1}c_k\bar{Z}$$

Then,

$$a_7^k = (1-e^{-rh})(1+c_k)\frac{N\alpha-1}{N-1} - \frac{(1-e^{-rh})\gamma_d}{r}(1+c_k)f_k + 2e^{-rh}a_4^{k+1}(1+c_k)f_k + e^{-rh}a_7^{k+1}(1+c_k)$$
$$= -\frac{f_k}{b_k(N-1)}(1+c_k)$$

and

$$a_8^k = \frac{c_k f_k}{b_k} + \left(\frac{f_k}{b_k(N-1)} - \frac{f}{b}\right)(1+c_k)$$

Adding the equations for a_7, a_8,

$$a_7^k + a_8^k = -\frac{f_k}{b_k} = (1-e^{-rh}) + e^{-rh}(a_7^{k+1} + a_8^{k+1}).$$

Then,

$$\begin{aligned}
a_9^k = \frac{f_k^2}{b_k} &- (1-e^{-rh})f_k\frac{N\alpha-1}{N-1} + (1-e^{-rh})f_k\frac{N(1-\alpha)}{N-1} + \frac{(1-e^{-rh})\gamma_d}{r}f_k^2 \\
&- 2e^{-rh}a_4^{k+1}f_k^2 - e^{-rh}a_7^{k+1}f_k + e^{-rh}a_8^{k+1}f_k + e^{-rh}a_9^{k+1} \\
&= \frac{f_k^2}{b_k} + \frac{f_k^2}{b_k(N-1)} + (1-e^{-rh})f_k\frac{N(1-\alpha)}{N-1} + e^{-rh}a_8^{k+1}f_k + e^{-rh}a_9^{k+1} \\
&= -\frac{(1-e^{-rh})\gamma_d f_k^2}{r} + \frac{2f_k^2}{b_k(N-1)} + 2e^{-rh}a_4^{k+1}f_k^2 + e^{-rh}a_9^{k+1} \\
&= \frac{2f_k^2}{b_k(N-1)} - \frac{c_k f_k^2}{b_k(1+c_k)(N-1)} + e^{-rh}a_9^{k+1} \\
&= \frac{(2+c_k)f_k^2}{b_k(1+c_k)(N-1)} + e^{-rh}a_9^{k+1}
\end{aligned}$$

Therefore, we have

$$a_0^k = -\bar{Z}^2 c_k^2\left(\frac{1}{b_k(N-1)} + e^{-rh}a_4^{k+1}\right) + e^{-rh}a_0^{t+1} + e^{-rh}a_5^{k+1}\sigma^2 + e^{-rh}a_6^{k+1}\sigma_N^2 + e^{-rh}a_9^{k+1}\sigma_N^2$$

$$a_1^k = \frac{c_k(c_k+1)\bar{Z}}{b_k(N-1)} - \frac{a_k + c_k\bar{Z}}{b_k}$$

$$a_2^k = f_k\frac{c_k\bar{Z}}{b_k(N-1)} - (1-e^{-rh})c_k\bar{Z}\frac{N\alpha-1}{N-1} + e^{-rh}a_2^{k+1} - e^{-rh}a_7^{k+1}c_k\bar{Z}$$

$$a_3^k = -\frac{c_k f_k N\bar{Z}}{b_k(N-1)} - (1-e^{-rh})c_k\bar{Z}\frac{N(1-\alpha)}{N-1} + e^{-rh}a_3^{k+1} - e^{-rh}a_8^{k+1}c_k\bar{Z}$$

$$a_4^k = -\frac{(1-e^{-rh})\gamma_d}{2r}(1+c_k)^2 + e^{-rh}a_4^{k+1}(1+c_k)^2$$

$$a_5^k = (1-e^{-rh})f_k\frac{N\alpha-1}{N-1} - \frac{(1-e^{-rh})\gamma_d}{2r}f_k^2 + e^{-rh}a_4^{k+1}f_k^2 + e^{-rh}a_5^{k+1} + e^{-rh}a_7^{k+1}f_k$$

$$a_6^k = -\frac{f_k^2}{b_k} - (1-e^{-rh})f_k\frac{N(1-\alpha)}{N-1} - \frac{(1-e^{-rh})\gamma_d}{2r}f_k^2 + e^{-rh}a_4^{k+1}f_k^2 + e^{-rh}a_6^{k+1} - e^{-rh}a_8^{k+1}f_k$$

$$a_7^k = -\frac{f_k}{b_k(N-1)}(1+c_k)$$

$$a_8^k = \frac{c_k f_k}{b_k} - \frac{f_k(N-2)}{b_k(N-1)}(1+c_k)$$

$$a_9^k = \frac{(2+c_k)f_k^2}{b_k(1+c_k)(N-1)} + e^{-rh}a_9^{k+1}$$