The Tradeoff between Discrete Pricing and Discrete Quantities: Evidence from U.S.-listed Firms

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Economists usually assume that price and quantity are continuous variables, while most market designs, in reality, impose discrete tick and lot sizes. We study a firm's trade-off between these two discretenesses in U.S. stock exchanges, which mandate a one-cent minimum tick size and a 100-share minimum lot size. A uniform tick size favors high prices because the bid–ask spread cannot be lower than one cent. A uniform lot size favors low prices because low prices reduce adverse selection costs for market makers when they have to display at least 100 shares. We predict that a firm achieves its optimal price when its bid–ask spread is two ticks wide, when the marginal contribution from discrete prices equals that from discrete lots. Empirically, we find that stock splits improve liquidity when they move the bid–ask spread towards two ticks; otherwise, they reduce liquidity. Liquidity improvements contribute 95 bps to the average total return on a split announcement of 272 bps. Optimal pricing can increase the median U.S. stock value by 69 bps and total U.S. market capitalization by $54.9 billion.

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1. Introduction

Price and quantity are two of the most important variables in economics. Most economists assume that price and quantity are continuous, but they are discrete in reality, even in most liquid markets like U.S. stock exchanges. No one can trade half a share, and a market maker generally must display a quote for one round lot of 100 shares; the minimum price variation (the tick size) is 1 cent for stocks priced above $1 per share. A U.S. firm can, therefore, choose a high price per share for a more continuous price but a more discrete quantity or a low price per share for a more discrete price but a more continuous quantity. In this paper, we show that the tradeoff between these seemingly small frictions plays a significant role in shaping the behavior of traders and listing firms.

[Insert Figure 1 about here]

Panel A in Figure 1 displays results implying that stocks are most liquid when their prices are neither too high nor too low. Panel B displays preliminary evidence that the trade-off between discreteness in quantity and discreteness in price drives this U-shaped pattern. Stocks with prices that are too low suffer from tick-size constraints. As the bid–ask spread cannot drop below 1 cent, the percentage bid–ask spread decreases with share prices for stocks for which the bid–ask spread is 1 cent, as indicated by the bottom left frontier in the figure. Stocks with prices that are too high suffer from lot-size constraints. Even for large firms such as Google and Amazon, the median depth at the National Best Bid and Offer (NBBO) and trade sizes are both exactly 100 shares. Once the lot size is binding, an increase in share prices amplifies the market maker's obligation to maintain round lots; the percentage bid–ask spread tends to increase with prices because market makers lose more money once they are adversely selected.

We discover a Two-Tick Rule for optimal pricing: Every firm reaches its optimal price when its bid–ask spread is two ticks. Intuitively, as the friction caused by a discrete price is one tick, a firm reaches its optimal price when the friction caused by a discrete lot is also one tick, i.e. when its bid–ask spread is two ticks. Our empirical results agree with the two-tick prediction. Stock splits improve liquidity if they move the nominal spread towards two ticks, whereas those that move it away from two ticks reduce liquidity. We find that most stock splits move the nominal spread towards two ticks and that they reduce the percentage bid–ask spread by 15.22 bps. The liquidity gains from stock splits generate a 95 bps increase in share prices. We estimate that the median U.S. stock value would increase by 69 bps if all firms were to move to their optimal prices.

Our model includes three types of players. In the first stage, the market designer chooses tick and lot sizes. To reflect the reality in the U.S., we focus on the system that imposes uniform tick and lot sizes, in which tick and lot sizes are the same for all stocks, and in Section 4 we compare the effects of uniform tick and lot sizes with those of proportional tick and lot sizes. In the second stage, a firm chooses its share price to maximize its liquidity as measured by the percentage spread. In the third stage, a market maker posts competitive bid prices to sell and ask prices to buy. The market maker earns the bid–ask spread if an uninformed trader hits her quotes, but she loses money when an informed trader adversely select the stale quotes.

We first consider cases where the market designer mandates a discrete quantity but keeps pricing continuous. A discrete lot size leads to the Square Rule: an H-fold reduction in share price leads to an H^2-fold reduction in the bid–ask spread. Therefore, a firm's percentage spread drops H-fold after an H-for-1 split. A lower price increases liquidity because of traders' interactions in the third stage. To minimize the loss from stale quotes, the market maker always chooses to display a minimum lot and then refills the lot once it is consumed. Uninformed traders also break their demand into a series of child orders of the minimum lot. As a consequence, the dollar size that a market maker has to display decreases linearly with the stock price. When a firm implements an H-fold reduction in its price, it creates an H-fold reduction in the loss for its market maker. In turn, the market maker can afford a percentage spread that is H times tighter or a bid–ask spread that is H^2 times tighter. A discrete quantity, therefore, favors a lower price per share.

A discrete price, however, favors a higher price per share. We first find that the bid–ask spread under discrete pricing is equal to the bid–ask spread under continuous pricing plus one tick. This result decomposes bid–ask spread s into two parts: a part Δ that is driven by a discrete price and a part $s-\Delta$ that is driven by a discrete quantity. An H-for-1 split reduces the lot-driven spread to $\frac{s-\Delta}{H^2}$, while the tick-driven spread remains at Δ. We then discover the Modified Square Rule: An H-for-1 split leads to a bid–ask spread of $\Delta+\frac{s-\Delta}{H^2}$. In turn, a firm can simply choose H, such that $\Delta+\frac{s-\Delta}{H^2}$ is equal to 2Δ, to achieve its optimal price. When $\Delta = 1$ cent, the optimal H is $\sqrt{s-1}$.

Our model generates three lines of predictions. First, for a given nominal price, the Modified Square Rule predicts a firm's liquidity, no matter whether the firm is at its optimal price. Our empirical results regarding stock splits show that a 1 bps increase in the percentage spread predicted by the Modified Square Rule leads to a 0.97 bps increase in the realized change in the percentage spread. In the cross-section, the Modified Square Rule explains 83% of the variation in the bid–ask spread with only the three variables modeled by our paper. We find that the lot-driven spread, which is equal to the bid–ask spread minus one tick, has the following relationship with the three independent variables. 1) The lot-driven spread decreases almost linearly with the dollar volume because doubling the dollar volume doubles the revenue base for market makers and cuts the lot-driven bid–ask spread by half. 2) The lot-driven spread increases linearly with volatility because the loss from stale quotes increases linearly with volatility. 3) An increase in the nominal price plays a role similar to that of volatility: as the nominal price doubles, the loss from being adversely selected also doubles. As an increase in the nominal price also causes a mechanical linear increase in the bid–ask spread, we find that the lot-driven spread follows a quadratic relationship with the nominal price.

Second, we predict an optimal price for each firm. The two variables in our model—volatility and dollar volume—explain 61% of the cross-sectional variation in the share price. We find that firms with higher volatility have lower prices, which is consistent with a finding reported in Baker, Greenwood, and Wurgler (2009), who find that volatile firms are more likely to split their stocks. Baker, Greenwood, and Wurgler (2009) characterize this result as a puzzle because volatile firms should have a weaker incentive to split because they have a "greater chance of reaching a low price anyway." Our model provides two interpretations of their puzzle. The first interpretation reflects the tick constraint. Stocks with higher volatility face higher adverse selection risk and thereby higher percentage bid–ask spreads. Higher percentage bid–ask spreads then relieve tick constraints: stocks with higher volatility reach their optimal two-tick bid–ask spreads at lower prices. The second interpretation reflects the lot constraint. A rise in volatility increases market makers' losses when they are adversely selected, and firms should choose lower prices to reduce the losses their market makers experience. Weld et al. (2009) use industry fixed effects to explain stock prices, and we find that the explanatory power of industry fixed effects is superseded by volatility. Therefore, firms in the same industry have similar nominal prices because they have similar volatility. Our model predicts that stocks with higher dollar volumes should have higher prices because an increase in the dollar volume reduces the percentage spread. Therefore, stocks with higher dollar volumes reach the optimal two-tick bid–ask spread at higher prices. Indeed, we find that stock prices increase with dollar volumes in our data.

Third, as firms can manage their prices through stock splits, our model offers rationales for stock splits and returns after stock splits. The modified square rule predicts that a split improves liquidity if the split moves a firm's nominal bid–ask spread closer to the two-tick optimum, otherwise the split reduces liquidity. We find that 1,077 out of 1,183 splits move nominal prices closer to the optimum. Among the 106 incorrect splits, 71 make the correct decision to split, except that they choose a split ratio that is overly aggressive. Overall, the percentage spread drops by 15.22 bps after splits. The effect on liquidity is so significant that it affects firm value: a 1 bps reduction in the predicted percentage spread increases firm value by 6.25 bps. As a consequence, the correct split ratios contribute 95 bps (6.25*15.22) to the average return on a split announcement of 272 bps.

Qualitatively, our empirical results indicate the stocks splits are rational response to tick and lot constraints. Further, our paper provides quantitative estimates about two questions. 1) What is the best ratio to maximize the benefits from splits? 2) How large are such benefits? A firm's optimal price depends on its characteristics, but managers do not need to calibrate these parameters to determine the optimal split ratio because a U.S. listed firm needs to know only its current spread s and then split at a ratio of $\sqrt{s-1}$. The modified square rule also offers an estimation of the benefit of a split that fits with the data well. For example, as of 2019 Amazon's stock was priced at $1,800 per share and its bid–ask spread was 60 cents (3.3 bps), while Apple's stock was priced at $200 per share and its bid–ask spread was 1.8 cents (0.9 bps). We find that the differences in nominal prices can explain the 3.7-times greater differences in transaction costs for two otherwise similar firms. If Amazon were to split 9-for-1, the modified square rule predicts that its nominal spread would become $\frac{60-1}{9^2}+1=1.73$ cents, which is similar to Apple's spread. Under a suboptimal price, Amazon holders paid $294 million in the (half) bid–ask spread in 2019, but Apple holders paid only $66 million. In fact, the optimal split ratio for Amazon is $\sqrt{59}$-for-1, which would save Amazon shareholders more than $230 million per year in transaction costs. Also, as an increase in liquidity increases share prices, Amazon's market cap would increase by $1.35 billion. The market value of U.S. firms would increase by $54.9 billion.

For firms, the tick and lot sizes are both exogenous, but regulators can change either of them. Our analysis of the market designer then provides insights into 1) the firm's best responses to changes in tick or lot sizes, 2) the impact of changing tick and lot sizes on liquidity, and 3) the comparison between the effects of uniform and proportional tick and lot sizes.

The following Square-Root Rule addresses the first two questions. When tick and lot sizes are uniform, a firm should respond to changes in tick and lot sizes in their square roots, and liquidity changes under such a best response account for the other square root. For example, in 2016 the SEC increased the tick size from one cent to five cents for 1,200 randomly selected firms. A firm's optimal response is a $\sqrt{5}$-for-1 reverse-split, because it maintains the same marginal contribution from the tick size ($\sqrt{5}$) and the lot size ($\sqrt{5}$). The firm's relative tick size and dollar lot size both increase by the square root $\sqrt{5}$. The Square-Root Rule leads to a spillover effect: a policy initiative that aims to make pricing more discrete would, in equilibrium, make quantity more discrete. The Two-Tick Rule still holds, but two ticks are now ten cents, leading to a $\sqrt{5}$ increase in the percentage spread. Thus, we encourage the SEC to consider reducing the tick size to improve market liquidity. Our model also provides support for the plan to reduce the lot size initiated by the Securities Information Processors committee.

In our model, the first best is continuous tick and lot sizes. We show that, once both have to be discrete, uniform tick and lot sizes dominate proportional tick and lot sizes. The uniform system seems more like a "one-size-fits-all" solution, but it enables a firm to choose an optimal price to balance discrete prices and quantities. The proportional system, however, destroys such flexibility. For example, consider a liquid $300 stock and an illiquid $3 stock. Both have chosen an equilibrium two-cent spread, one cent from the tick and one cent from the lot. A proportional system based on a \$30 stock will assign a ten times larger tick size and a ten times smaller lot size to the \$300 stock, leading to a 10.1 cent spread ($= 10 + 0.1$). On the other hand, the \$3 stock also loses its optimal tick-lot balance and achieves a bid–ask spread of 10.1 cents ($= 0.1 + 10$). Therefore, the proportional system harms liquidity because it is a true one-size-fits-all system that enforces a uniform level of discreteness on stocks with heterogeneous characteristics.

As the first study to examine discreteness in both price and quantity, our paper offers significant new insights compared with models where one variable is continuous. Angel (1997) argues that a fivefold increase in the tick size would have no economic impact because firms can perfectly neutralize such a policy change through a 5-for-1 reverse split. By adding discrete quantities, we find that a 5-for-1 reverse split does not neutralize a fivefold increase in the tick size. More surprisingly, a 5-for-1 reverse split leads to the same outcome as doing nothing at all. A 5-for-1 reverse split retains the original relative tick size, but increases the dollar lot size by a factor of five over the original size. As the total percentage spread is the sum of tick- and lot-driven components, a 5-for-1 reverse split leads to the same outcome as doing nothing.

Budish, Cramton, and Shim (2015, BCS hereafter), who consider a market with discrete quantities but continuous pricing, find that public information leads to a positive percentage spread. We find that this percentage spread is a linear function of lot size: if the lot size decreases to one-half of a share, the percentage spread also decreases by one-half. Therefore, the percentage spread converges on zero when the lot size converges on zero. A firm can also reduce the dollar lot size, the loss to its market maker, and the percentage spread through aggressive stock splits. The economic force that counterbalances aggressive splits is discrete pricing.

Our paper answers two main research questions in the literature on stock splits. 1) Why do firms split their stocks? 2) What explains the positive returns after splits?

Our answers differ from two benchmarks: the signaling channel and the liquidity channel. In the signaling channel, the cost of a signal comes from reduced liquidity (Brennan and Copeland, 1988), yet we find an increase in liquidity in our sample. Also, Fama et al. (1969), Lakonishok and Lev (1987), and Asquith, Healy, and Palepu (1989) find that firms' earnings, profits, and stock prices increase significantly before splits but not after splits. Their results do not support the signaling channel but do support our tick-and-lot channel. A previous increase in a stock price increases the lot constraint on that stock, and stock splits are the best response to lot constraints. In the liquidity channel (Lamoureux and Poon, 1987 and Maloney and Mulherin, 1992), firms use stock splits to attract retail traders, and an increase in the number of uninformed traders increases volume and liquidity, yet we find that institutional holdings increase after splits in our sample.

We provide the best fit for cross-sectional variation in liquidity to date, even though we use only a subset of the explanatory variables in existing benchmarks (Stoll 2000; Madhavan 2000). This surprising increase in the R-squared with fewer variables stems from two economic forces. First, our model provides a better functional form. Madhavan (2000) uses $price^{-1}$ as the control variable while Stoll (2000) uses the log (price). Both of these functional forms impose a monotonic relationship between the nominal price and the percentage spread, but we find that their true relationship is U-shaped. We subtract one tick from the bid–ask spread to control for the tick-driven spread and then add the log of the price to control for the lot-driven spread.

Second, our model helps to rule out redundant variables that have been included in previous specifications. For example, all existing specifications control for the market cap, following the intuition that large stocks are more liquid. Our model suggests that the market cap affects liquidity only through its impact on the dollar volume. Holding the share turnover rate fixed, a large-cap stock has a lower percentage spread because it has a greater dollar volume. A small firm with higher turnover should, however, be as liquid as a large firm with lower turnover as long as they have the same dollar volume. Therefore, the dollar volume absorbs the impact of the market cap and turnover on liquidity. Our interpretation addresses a puzzle raised in Stoll (2000), who finds that the regression coefficient before the market cap is not always positive after controlling for the dollar volume. The 83% R-squared in our regression suggests that future empirical research may use our specification as benchmark against which to evaluate additional variables that can potentially explain stock market liquidity.

In time series, we provide the first unified explanation of four salient facts that go hand in hand after trading became automated: a reduction in the bid–ask spread (Hendershott, Jones, and Menkveld, 2011), the decline in depth towards one lot (Angel, Harris, and Spatt, 2015), the dominance of trades of one lot (O'Hara, Yao, and Ye, 2014), and the proliferation of algorithmic traders who are not as fast as high-frequency traders (HFTs) (O'Hara, 2015). We are able to generate all these predictions in one model because we model interactions between multiple types of algorithmic traders, whereas most studies include at most one type of algorithmic trader: HFTs. Note that liquidity demanders' execution algorithms do not need to execute as quickly as those of HFTs. They just need to be fast enough to slice and dice large latent demand into a series of child orders of one lot apiece. Along with the reduction in the bid–ask spread, our model also explains the depth of one lot and a trade size of one lot.

2. A Continuous-Pricing Model

In this section we consider the equilibrium outcome where the market designer mandates a discrete lot size $L \geq 0$ but the price remains continuous.

The firm's fundamental value v_t evolves as a Poisson jump process at an arrival rate of λ_J, where t runs continuously on $[0, \infty)$. The jump sizes are $\sigma\%$ or $-\sigma\%$ of

v_t with equal probability. The firm decides its nominal price $p_t := \frac{v_t}{h_t}$ with the aim of minimizing its percentage spread $\mathcal{S}_t := \frac{s_t}{p_t}$, where h_t is shares outstanding.

Given p_t and L, the market maker set competitive bid prices to sell and competitive ask prices to buy. The bid-ask spread s_t equalizes the revenue gain from uninformed traders and the loss incurred from the informed traders. Uninformed traders arrive at the market at Poisson intensity λ_I. Each uninformed trader has an inelastic need to buy or sell the stock at equal probability. For simplicity, we normalize the need of each uninformed trader as v_t, so that λ_I is the turnover rate per unit-time, and $\lambda_I v_t$ is the trading volume from uninformed traders per unit time. We call v_t the size of the parent order. As $p_t L$ represents dollar lot size, the parent order is equal to $\frac{v_t}{p_t L}$ lots. The uninformed traders' objective function is to minimize the transaction cost of the parent order by choosing the way to slice the parent order into a series of child orders. For example, one feasible strategy is to separate her demand into $\frac{v_t}{p_t L}$ child orders of one minimum lot apiece. We assume that L is small enough so that the parent order is divisible to round lot orders. [1]

Informed traders aim to adversely select the market maker during value jumps. There are two ways to interpret adverse selection risk in our model. First, v_t is common knowledge but the market maker may fail to cancel stale quote. In this case, the market maker in our model is equivalent to the liquidity-providing HFT in BCS, and informed traders are equivalent to the stale-quote-sniping HFTs in BCS. Second, v_t is private information, but is revealed after each trade (Aquilina, Budish, and O'Neill, 2020; Baldauf and Mollner, 2020; Admati and Pfleiderer 1988). Both scenarios lead to the same model. For tractability, we assume that informed traders

[1] Institutional traders' liquidity demand is much greater than the size of child orders and the lot size. For example, O'Hara (2015) finds that a volume-weighted average price (VWAP) algorithm, on average, turns each parent order into 55,235 child orders.

can adversely select the market maker only once per piece of information. Without this simplification, the optimization problem for the firm is not well-defined because the bid–ask spread, the objective that the firm aims to minimize, would be a nonstationary function over time.[2]

2.1 Traders' Choice

We solve the model through backward induction. Proposition 1 presents the optimal choice of the market maker and uninformed traders given p_t and L.

Proposition 1. (Continuous-Price Bid–ask spread) *With zero tick size and lot size L, the equilibrium percentage bid–ask spread is $\mathcal{S}_t^L = \frac{2\sigma\lambda_J p_t L}{\lambda_I v_t + \lambda_J p_t L}$:*

(i) The depth at the Best Bid and Offer (BBO) is exactly L shares, and the market maker refills L shares at the BBO immediately after each trade.

(ii) Uninformed traders slice their demand into a series of child orders. Each child order includes exactly L shares.

Proposition 1 shows that the market maker never displays more than the minimum lot at the BBO, and uninformed traders never take more than the minimum lot per trade. Suppose that the market maker quotes a second lot at BBO. Her loss during value jump doubles when she was adversely selected. Therefore, the market maker's average quote for two minimum lots are worse than the quote for one minimum lot. In turn, uninformed traders should divide their demand into minimum lots and the market maker quickly refills the minimum lot at the BBO once it is consumed. These two predictions match well with stylized facts. O'Hara, Yao, and Ye (2014) find that more than 50% of trades involve exactly 100 shares,

[2] Glosten and Milgrom (1985), Vayanos (1999), and Back and Baruch (2004) characterize these non-stationary bid–ask spreads, though their solutions are either very complicated or available only numerically.

and we find that this ratio is as high as 87.5% for stocks that are not bound by the tick size. As shown in Panel B of Figure 1, most stocks that are not bound by the tick size have a depth of exactly one lot.

The equilibrium spread equates the revenue from the bid-ask spread and the cost from adverse selection. Denote $V = \lambda_I v_t + \lambda_J p_t L$ as the total dollar volume per unit of time. $\lambda_I v_t$ comes from uninformed traders whereas $\lambda_J p_t L$ comes from informed traders. The revenue for the market maker per unit of time is $V \cdot \frac{\mathcal{S}_t^L}{2}$. The loss from adverse selection is $\sigma \cdot p_t L \cdot \lambda_J$: σ is the percentage loss, $p_t L$ is the base for the percentage loss and λ_J is the arrival rate of the loss. Equating the revenue and cost, we have

$$(\lambda_I v_t + \lambda_J p_t L) \cdot \frac{\mathcal{S}_t^L}{2} = \sigma \cdot p_t L \cdot \lambda_J. \tag{1}$$

Thus,

$$\mathcal{S}_t^L = \frac{2\sigma\lambda_J p_t L}{\lambda_I v_t + \lambda_J p_t L}. \tag{2}$$

$\mathcal{S}_t^L$ decreases strictly with the dollar lot size $p_t L$, because a reduction in the dollar lot size reduces the market maker's loss when she is adversely selected. The market maker can still accommodate demand from uninformed traders with more trades and a small dollar lot size.

Enjoying the benefit of slicing orders to the minimum lot, however, requires technology that makes it possible to slice the parent order into many child orders. Therefore, our model rationalizes algorithmic trading who are slower than HFTs. [3] Brogaard et al. (2015) document the existence of "SlowColos" who co-locate at a stock exchange but whose speed technologies are inferior to those of HFTs. Yet it is unclear why they neither choose to be the fastest nor choose to be slow. We

[3] To the best of our knowledge, the only other interpretation has been offered in a companion paper (Li, Wang, and Ye 2020), which shows that slow traders use execution algorithms to choose between market and limit orders.

conjecture that those SlowColos are algorithms aiming at order executions. These execution algorithms need to be fast enough to slice a larger number of child orders in a short time, but they do not need to be the fastest to reduce adverse selection risk or to adversely select slow traders.

Proposition 1 pertains to the percentage spread, while Corollary 1 shows the square rule that applies to the bid–ask spread.

Corollary 1 (The Square Rule). *Under continuous pricing, while controlling for trading volume and volatility, the nominal bid–ask spread* $s_t^L = \frac{2\sigma\lambda_J p_t^2 L}{V}$ *is proportional to the square of the nominal price.*

An increase in p_t would increase the bid–ask spread in square after controlling for volume and volatility, because an increase in p_t first leads to a mechanical linear increase in the bid–ask spread while holding the percentage spread fixed, and an increase in p_t also increases the percentage spread linearly as a result of an increase in adverse selection costs.

2.2 The Firm's Choice

When pricing is continuous, the firm's decision is simple: it chooses $p_t \to 0$ such that $\mathcal{S}_t$ converges on 0. By choosing a very low price, the firm makes its adverse selection cost and bid–ask spread negligible. Lot constraints, therefore, favor low prices. When the lot size is the only friction, firms will split their stocks aggressively to minimize such friction.

By considering the lot size, our paper makes two contributions relative to BCS. The bid–ask spread in BCS is positive because 1) the lot size is binding at "one share" and 2) the nominal price of the stock is equal to v_t. Our paper points out two alternative solutions to the BCS problem if the price is continuous. The policy solution is to reduce the lot size, which we will discuss in greater depth in Section

4.1. The market solution is to allow a firm to reduce its nominal prices. The constraint that prevents the firm from choosing very low prices comes from the other friction: discrete pricing. We consider the tradeoff between discrete pricing and quantity in the next section.

3. The Model with Discrete Pricing

In this section, the market designer mandates a discrete tick size Δ in addition to the discrete lot size L, and trades and quotes occur only at the pricing grid $\{\Delta, 2\Delta, 3\Delta, \ldots\}$. Then, a firm cannot reduce its bid–ask spread below one tick no matter how aggressively it splits its stock. Splits may increase the percentage spread if a firm's bid–ask spread is close to one tick. The tick constraint, therefore, favors high prices. We solve the model through backward induction. In Subsection 3.1, we analyze traders' decisions given p_t and L, the main purpose of which is to quantify the frictions generated by the tick size. In Subsection 3.2, the firm chooses the optimal nominal price, which balances the frictions generated by lot and tick sizes.

3.1. Traders' Decisions and the Friction from the Tick Size

In this subsection we analyze the traders' decisions, taking p_t and L as given. Under discrete pricing, the market maker can no longer quote competitive prices at $p_t \pm \frac{s_t^L}{2}$ unless those prices coincide with a tick grid. Proposition 2 shows that she quotes a bid price at the tick immediately below $p_t - \frac{s_t^L}{2}$ and an ask price at the tick immediately above $p_t + \frac{s_t^L}{2}$.

Proposition 2 (Discrete Price Bid–ask spread). *With a tick size* Δ:

(i) The competitive market maker quotes an ask price, $A_t = \left\lceil \frac{p_t + s_t^L/2}{\Delta} \right\rceil \Delta$,

and a bid price, $B_t = \left\lfloor \frac{p_t - s_t^L/2}{\Delta} \right\rfloor \Delta$*, where* $\lceil x \rceil$ *is the smallest integer larger than* x*, and* $\lfloor x \rfloor$ *is the largest integer smaller than* x*.*

(ii) The bid-ask spread is $W_t = B_t - A_t - s_t^L \in [0,2\Delta)$ *wider than the spread* s_t^L *under continuous pricing.*

(iii) Define $z_t := \{\frac{p_t}{\Delta}\}$*, where* $\{x\}$ *is the fractional part of* x*. As long as* $p_t \gg \Delta$ *and with long enough evolving time* $t \gg \frac{1}{\lambda_J}$*, we have* $z_t = \{\frac{p_t}{\Delta}\} \xrightarrow{d} U[0,1)$ *and the* $\mathbb{E}(W_t) = \Delta$ *for any initial* p_0*. Therefore, the bid–ask spread under discrete pricing is one tick wider than* s_t^* *under continuous pricing:*

$$s_t^{tot} = s_t^L + \Delta = \frac{2\sigma\lambda_J p_t^2 L}{\lambda_I v_t + \lambda_J p_t L} + \Delta. \tag{3}$$

Proposition 2 shows that the tick size widens the spread unless both the bid and ask prices coincide with a tick grid. The tick size can widen the bid–ask spread by at most two ticks: one tick on the ask side and another on the bid side. The exact size of the widening is a random variable that depends on p_t and s_t^L. Fortunately, Corollary 2 shows that the expectation of the widening effect is one tick. The intuition behind this is that, as time goes to infinity, p_t would not cluster at any sub-tick locations. For a Poisson jump process, the residual $\left\{\frac{p_t}{\Delta}\right\}$ tends to uniformly distribute within the tick.[4] Therefore, the bid–ask spread under a discrete pricing, s_t^{tot}, is equal to the bid–ask spread under continuous pricing plus one tick.

Equation (3) decomposes the bid–ask spread into a lot-driven component and a

[4] The only exception is when $\frac{p_t}{\Delta} \to 0$, i.e. when the nominal price of the stock is less than one tick. In this case, $z_t = \{\frac{p_t}{\Delta}\}$ may cluster around 0. In reality, both the NYSE and NASDAQ will delist a stock if its price falls under \$1.00 (i.e. when $p_t < 100\Delta$). Therefore, $p_t \gg \Delta$ generally holds.

tick-driven component. The lot-driven component, $\frac{2\sigma\lambda_J p_t^2 L}{\lambda_I v_t+\lambda_J p_t L}$, follows the Square Rule. The tick-driven component is always equal to Δ, the expectation of the widening effect. Therefore, an increase in p_t dilutes the widening effect proportionately.

3.2. The Firm's Decision and the Optimal Nominal Price

The firm chooses p_t to minimize the percentage spread, given lot size L and tick size Δ. By choosing p_t, the firm balances the friction caused by the discrete price as measured by the relative tick size $\frac{\Delta}{p_t}$, and the friction caused by the discrete quantity as measured by the size of the dollar lot $p_t L$. To obtain the object function, divide both sides of Equation (3) by p_t:

$$\min_{p_t} \mathcal{S}_t^{tot} = \frac{S_t^{tot}}{p_t} = \frac{2\sigma\lambda_J p_t L}{\lambda_I v_t+\lambda_J p_t L} + \frac{\Delta}{p_t}. \tag{4}$$

The first term in the objective function, $\frac{2\sigma\lambda_J p_t L}{\lambda_I v_t+\lambda_J p_t L}$ is driven by the lot size and it increases with p_t. The second term, $\frac{\Delta}{p_t}$, is the relative tick size and it decreases with p_t. The first-order condition for equation (4) is:

$$\frac{\partial \mathcal{S}_t^{tot}}{\partial P_t} = \frac{2\sigma\lambda_J L\lambda_I v_t}{(\lambda_I v_t+\lambda_J p_t L)^2} - \frac{\Delta}{p_t^2} = 0, \tag{5}$$

or

$$\frac{2\sigma\lambda_J L\lambda_I v_t}{(\lambda_I v_t+\lambda_J p_t L)^2} = \frac{\Delta}{p_t^2}. \tag{6}$$

Equation (6) shows that the optimal nominal price for a firm, p_t^*, should equalize the marginal contribution from tick size $\frac{\Delta}{p_t^2}$ to the marginal contribution of lot size $\frac{2\sigma\lambda_J L\lambda_I v_t}{(\lambda_I v_t+\lambda_J p_t L)^2}$. If a firm chooses a nominal price that is higher than p_t^*, a small decrease in p_t increases the percentage spread generated by the tick size but reduces the percentage spread generated by the lot size by a larger amount. Therefore, a firm should split its stock when its price is too high. Similarly, when $p_t < p_t^*$, the marginal contribution from the tick size is larger than the marginal contribution from the lot size. A firm should then reverse-split its stock to reduce price discreteness and increase quantity discreteness.

Proposition 3 provides the solution that determines the optimal price. The solution becomes very intuitive when the latent demand of uninformed traders is much larger than the lot size, that is, $L \to 0$.

Proposition 3 (Golden Rule of Two Cents). *When the tick size is Δ and the lot size is L, the optimal nominal price is $p_t^* = \underset{p_t}{argmin}\, \mathcal{S}_t^{tot} = (\sqrt{\frac{2\sigma\lambda_J L}{\lambda_I \Delta v_t}} - \frac{\lambda_J L}{\lambda_I v_t})^{-1}$ and $s_t^{tot}(p_t^*) = \Delta + \Delta/(1 - \sqrt{\frac{\Delta\lambda_J L}{2\sigma\lambda_I v_t}})$. When $L \to 0$, $s_t^{tot} = s_t^L + \Delta \approx \frac{2\sigma\lambda_J p_t^2 L}{\lambda_I v_t} + \Delta$ and $\mathcal{S}_t^{tot} = \frac{2\sigma\lambda_J p_t L}{\lambda_I v_t} + \frac{\Delta}{p_t}$. The optimal nominal price that minimizes a firm's percentage bid–ask spread is $p_t^* = \underset{p_t}{argmin}\, \mathcal{S}_t^{tot} = \sqrt{\frac{\lambda_I \Delta v_t}{2\sigma\lambda_J L}}$, and $s_t^{tot}(p_t^*) = 2\Delta$.*

Proposition 3 offers a simple rule of thumb a firm can follow to determine its optimal price: a firm reaches its optimal nominal price when the nominal spread is two ticks. Stocks trading with a bid–ask spread that is smaller than two ticks are more tightly constrained by the tick size, and those firms can infer that their nominal prices are too low. Firms trading with a bid–ask spread wider than two ticks are more lot-bound and their nominal prices are too high.

The Two-Tick rule implies heterogeneity in nominal prices. For example, an increase in volatility, caused either by an increase in jump size σ or an increase in jump frequency λ_J, reduces the optimal nominal price. The intuition behind this is as follows. Holding all other things equal, an increase in volatility increases adverse selection risk and the percentage spread. Holding the nominal price fixed, an increase in the percentage bid–ask spread relieves the tick-size constraint.

Therefore, a firm should reduce its nominal price, increase the marginal contribution from the tick size, and decrease the marginal contribution from the lot size. A stock with a higher jump size σ, a higher jump-arrival rate λ_J, or a lower λ_I tends to realize a higher percentage spread, and that stock achieves a two-tick nominal spread at a lower nominal price than a stock with lower σ, lower λ_J, or higher λ_I. In summary, optimal pricing implies that the nominal spread is always two ticks, but the percentage spread under the optimal price depends on firm fundamentals.

An elegant feature of our model is that a firm does not need to calibrate σ, λ_J, and λ_I to estimate its optimal price, because a firm's current spread s_t^{tot} provides sufficient statistics for the decision. Define $H_t := \frac{p_{t-0}}{p_{t+0}}$ as the split ratio of a firm, i.e. the ratio of the nominal prices immediately before and after a split. When a firm splits by H_t-for-1, the tick-driven part remains unchanged at Δ while the lot-driven part follows the Square Rule and changes to $(s_t^{tot} - \Delta)/H_t^2$. Then, the total spread after splits becomes $(s_t^{tot} - \Delta)/H_t^2 + \Delta$. To achieve the optimal nominal spread of two ticks, a firm should choose an H_t such that

$$(s_t^{tot} - \Delta)/H_t^2 + \Delta = 2\Delta. \tag{7}$$

The solution for equation (7) leads to the optimal split ratio

$$H_t^* = \sqrt{\frac{s_t^{tot} - \Delta}{\Delta}}. \tag{8}$$

This Modified Square Rule provides a convenient way to determine the split/reverse split ratio that enables a firm to reach the two-tick spread.

Corollary 2 (The Modified Square Rule and the Optimal Split Ratio). *When the parent order is much higher than the lot size, an H_t-for-1 split changes the spread from s_t^{tot} to $(s_t^{tot} - \Delta)/H_t^2 + \Delta$, and the percentage spread changes by $R = \frac{(s_t^{tot} - \Delta)/H_t^2 + \Delta}{v_t/H_t} - \frac{s_t^{tot}}{v_t}$. R reaches its minimum if and only if $H_t^* = \sqrt{\frac{s_t^{tot} - \Delta}{\Delta}}$.*

In corollary 2, we modify the Square Rule in Corollary 1 to accommodate discrete pricing. The Modified Square Rule predicts the percentage spread change for any H_t, even if the H_t is not optimal. We test the Modified Square Rule in Section 5.

4. Policy Implications for Tick and Lot sizes

In this section, we allow the market designer to change tick and lot sizes and consider our model's policy implications. In Subsection 4.1, we consider the firm's responses to changes in tick and lot sizes. In Subsection 4.2, we show that a uniform tick- and lot-size system dominates a system with proportional tick and lot sizes.

4.1. Changes in a Uniform Tick- and Lot-Size Regime

Proposition 4 shows the firm's best response to changes in tick and lot sizes as well as the resultant change in liquidity.

Corollary 3. (The Square Root Rule) *When the lot size is small, a firm with* $p_t^* = \sqrt{\frac{\lambda_I \Delta v_t}{2\sigma\lambda_J L}}$ *should respond to tick- and lot-size changes by* $\sqrt{\Delta/L}$*. The nominal spread under optimal pricing,* $s_t^{tot}(p_t^*)$*, equals* 2Δ *regardless of* L *and* Δ*, and the smallest achievable percentage spread* $\mathcal{S}_t^{tot}(p_t^*) = \frac{2\Delta}{p_t^*} = \sqrt{\frac{8\sigma\lambda_J\Delta L}{\lambda_I v_t}}$ *is proportional to* $\sqrt{\Delta L}$*.*

Corollary 3 shows a series of responses in square roots after the market designer changes the tick or lot size. First, a firm's optimal response to a change in the tick or lot size is the square root of the change. For example, if regulators increase the tick size from one cent to five cents, firms should reverse-split their stocks by $\sqrt{5}$. This reverse-split ratio is optimal because it equals the marginal contribution from the tick size and the marginal contribution from the lot size. Both the relative tick size and the dollar lot size increase at the rate of the square root.

Second, liquidity also changes at the rate of the square root. To see this, recall that the optimal 1-for-$\sqrt{5}$ reverse split increases the lot-driven spread to 5 ($= \sqrt{5} \times \sqrt{5}$) cents, and the tick-driven spread is still 5 cents. The 1-for-$\sqrt{5}$ reverse split restores the two-tick optimal spread, except that the two ticks now equal ten cents. The optimal bid–ask spread increases fivefold and the nominal price increases by a factor of $\sqrt{5}$, leading to a $\sqrt{5}$-fold increase in the percentage spread. In summary, the Two-Tick rule always hold, but the firm's optimal response, the lot-driven percentage spread, the tick-driven percentage spread, and the total percentage spread all change in accordance with the Square Root Rule.

The same intuition applies to a reduction in the lot size. In 2019, the SIP Operating Committee solicited comments for a policy initiative designed to reduce the friction associated with odd-lot trades, or orders involving fewer than 100 shares. Stock exchanges and institutional traders proposed a more aggressive plan: reduce the threshold of the round lot to fewer than 100 shares.[5] Proposition 4 indicates that a reduction in the lot size improves liquidity, and firms should reverse-split their stocks to take full advantage of the benefit. For example, if the SIP committee reduces the round lot from 100 shares to 1 share, firms should reverse-split at a ratio of 1-for-$\sqrt{100}$ to maximize the benefit of the lot-size reduction. Such a reduction in the spread also explains why broker-dealers, who often provide execution within the bid–ask spread against retail traders (Boehmer et al., 2020), oppose the reduction in the official lot size. For them, a reduction in the lot size reduces the reference bid–ask spread in stock exchanges and thereby forces them to offer better

[5] NASDAQ's comment letter (https://www.theice.com/publicdocs/SIP_Comment_Nasdaq_redacted.pdf) suggests that "high value quotations with significant price discovery information would be protected, even if they were less than 100 shares." Citadel and Blackrock also support lot-size reduction in their comment letters. However, retail broker-dealers such as TD Ameritrade oppose the idea because "display of unprotected quotes will cause confusion and mistrust in the market."

prices to retail traders.

Corollary 3 shows that a policy initiative that aims to make pricing more discrete also makes quantity more discrete in equilibrium and vice versa. To the best of our knowledge, we are the first to identify this spillover effect. Angel (1997) considers only discrete pricing, and he finds that a 1-for-5 reverse split would neutralize a fivefold increase in the tick size. When we add discrete quantities, a 1-for-5 reverse split no longer neutralizes the increase in the tick size nor is the best response, and the 1-for-5 reverse split is as bad as doing nothing at all! The intuition behind this is as follows. Although a 1-for-5 reverse split restores the relative tick size, such aggressive reverse splits cause a fivefold increase in the dollar lot size. In equilibrium, a fivefold increase in the dollar lot size leads to the same increase in the percentage spread as a fivefold increase in the tick size does. For example, consider a firm that currently has an optimal spread of two cents; one cent comes from the tick size and one cent comes from the lot size. An increase in the tick size from one cent to five cents raises the spread from the tick size to five cents, leading to a nominal spread of six cents, which is three times the previous level. After a 1-for-5 reverse split, the tick-driven spread remains at five cents. A fivefold increase in the lot size raises the lot-driven spread to 5^2. The nominal spread now becomes $5 + 5^2$. After adjusting for the increase in the nominal price, the percentage bid–ask spread still increases by a factor of three ($= \frac{25+5}{2\times 5}$). In conclusion, a reverse split at the same rate as the increase in the tick size is as bad as doing nothing at all. Compared with the optimal response under the Square Root rule, the percentage spread increases by $\frac{3}{\sqrt{5}} \approx 1.34$ times.

4.2 Proportional vs. Uniform Tick and Lot Sizes

One plan for changing the lot size is to make it a function of price, such that high-priced stocks have smaller lot sizes.[6] This plan essentially generates a proportional lot size, which leads to a uniform dollar lot size for all stocks. Also, in many European countries, Hong Kong and Japan, the tick size increases with stock prices, which generally leads to a proportional tick size (a uniform relative tick size). Corollary 4 shows that if the tick size is proportional, firms should split their stocks to minimize friction from the lot size. On the other hand, if the lot size is proportional, firms should reverse-split to minimize the tick size friction. If both the lot and tick size are proportional, the choice of a nominal price becomes irrelevant. In this case, firms lose flexibility in the tick–lot balance, and almost all firms face worse liquidity.

Corollary 4. (Proportional Tick and Lot Systems) *(1) With fixed* Δ *and proportional lot size* $\mathbb{L}(p_t) = k^L/p_t$, *where* k^L *is a constant, the firm's optimal choice is* $p_t^* \to \infty$ *and* $\mathcal{S}^* = \frac{2\sigma\lambda_J}{\lambda_I v_t} k^L$*.(2) With fixed* L *and proportional tick size* $\mathbb{D}(p_t) = k^\Delta p_t$, *where* k^Δ *is a constant, the firm's optimal choice is* $p_t^* \to 0$ *and* $\mathcal{S}^* = k^\Delta$. *(3) With proportional tick* $\mathbb{D}(p_t) = k^\Delta p_t$ *and lot* $\mathbb{L}(p_t) = k^L/p_t$, $\mathcal{S} \equiv \frac{2\sigma\lambda_J}{\lambda_I v_t} k^L + k^\Delta$ *for any* p_t. *Adopting the proportional system in (3) with any reference price* p_Ω *such that* $k^\Delta = \Delta/p_\Omega$ *and* $k^L = Lp_\Omega$ *reduces liquidity for any stock with* $p \neq p_\Omega$.

In Table 1 we summarize the results of our discussions of the uniform or proportional tick size. As either k^Δ or k^L can equal 0, Table 1 also provides the results for continuous versus discrete tick sizes. The first best in our model is continuous tick and lot sizes. In this case, the bid–ask spread becomes 0. If one

[6] For example, Blackrock (2019) "believes that a more elegant solution for the inclusion of odd lots would be to move from a 'one-size-fits-all' approach to a multi-tiered framework where round lot sizes are determined by the price of a security." NASDAQ (2019) suggests "establishing a dollar threshold for the value of quotes to be protected, defined as price multiplied by the number of shares."

variable is continuous and the other is discrete, firms would split/reverse split to minimize the friction from the only discrete variable. A proportional tick size leads firms to choose low prices, because lowering prices reduces the lot-driven percentage spread without increasing the tick-driven percentage spread. The optimal percentage spread equals the mandated proportional tick size. A proportional lot size leads firms to choose high prices. Firms reverse-split to minimize the impact a discrete pricing. The optimal percentage spread contains only the lot-driven component, which is the same for all stocks under proportional lot sizes.

[Insert Table 1 about Here]

The most interesting contrasts appears at the diagonal, where we compare uniform tick and lot sizes with proportional tick and lot sizes. Such a comparison depends on k^{Δ} and k^{L}. A natural way to choose k^{Δ} and k^{L} is to use a representative stock. For example, a market designer can choose k^{Δ} and k^{L} such that the proportional tick and lot sizes for a \$30 benchmark stock do not change. Corollary 4 shows that such proportional systems would decrease liquidity for *all* stocks except the representative stock. The greater the distance between the stock price and the benchmark price, the greater the liquidity reduction.

For example, a proportional system can maintain the tick and lot sizes for a representative stock that is trading at \$30. The proportional system would impose a tenfold larger tick size and a 0.1-fold larger lot size on a \$300 stock. If the \$300 stock was at its equilibrium with a two-cent bid–ask spread, its tick-driven spread increases to ten cents and its lot-driven spread reduces to 0.1 cent, leading to an increase of the total spread from two cents to 10.1 cents. Symmetrically, the proportional system would impose a 0.1-fold larger tick size and tenfold larger lot size for a \$3 stock. If the \$3 stock currently trades with a two-cent bid–ask spread, its tick-driven spread would drop to 0.1 cent but its lot-driven spread would increase to 10 cents, leading to an increase in the total spread from two cents to 10.1 cents.

Under uniform tick and lot sizes, the firm choosing a \$300 (\$3) price is more (less) liquid than a firm choosing a \$30 price, but adopting a proportional tick and lot system reduces liquidity for both the \$300 and the \$3 stock at the same magnitude. Corollary 4 implies that, if regulators want to switch from a uniform system to a proportional system, they should not use any existing stock as the benchmark to set proportional tick and lot sizes.

The uniform system dominates the proportional system because the former is more flexible. The uniform system may seem less flexible because it mandates the same tick and lot size for stocks with varying prices. Yet the uniform system actually gives firms flexibility to choose the optimal balance between lot and tick sizes by adjusting nominal prices. More liquid stocks endogenously choose higher prices (i.e. higher dollar lot sizes and lower relative tick sizes), because the main friction comes from discrete pricing. Less liquid stocks endogenously choose lower prices (i.e. lower dollar lot sizes and higher relative tick sizes), because the main friction comes from trading large lots. Therefore, although the prefect system would utilize continuous tick and lot sizes, the uniform system at least offers one degree of freedom. The proportional system offers zero degrees of freedom because it mandates the same level of discreteness in price and quantity for stocks with heterogeneous characteristics. Other than the benchmark stock, which happens to fit the imposed tick and lot sizes, the proportional system reduces the liquidity of all other stocks. The biggest victims of the move from a uniform system to a proportional system would be stocks whose nominal prices (and implicitly stock characteristics) are most different from the benchmark stock.

5. Cross-sectional Tests

In this section, we test our model in the cross-section. In Section 5.1 we show that the cross-sectional variation in the bid–ask spread follows the Modified Square Rule quantitatively. Section 5.2 shows that our model also explains 61% of the variation in the nominal price.

5.1 A three-factor empirical model of liquidity

Formula (3) implies a three-factor model of cross-sectional variation in the bid–ask spread. To see that, if we multiply the denominator and numerator of the right-hand side of Formula (3) by Lp_t, we obtain

$$s_t^{tot} - \Delta = \frac{2L\sigma\lambda_J p_t^2}{\lambda_I v_t + \lambda_J p_t L} = \frac{2L\sigma\lambda_J p_t^2}{V} \quad (9)$$

Taking the natural log on both sides, we obtain

$$log(s_t^{tot} - \Delta) = 2log(p_t) - log(V) + log(\sigma\lambda_J) + const. \quad (10)$$

The empirical proxy for p_t is the nominal price, V is the dollar trading volume of the stock, and the proxy for $\sigma\lambda_J$ is stock volatility. Writing (11) into the form of an OLS test:

$$log(s_t^{tot} - \Delta)_i = \delta \cdot log(Price)_i + log(Volatility)_i + log(Volume)_i + \varepsilon_i. (11)$$

We predict that $\delta = 2$. The null hypothesis is $\delta = 1$: when the lot size does not impose a binding constraint on the bid–ask spread, $s_t^{tot} - \Delta$ should increase in one-to-one fashion with respect to the price.

We use daily Trade and Quote (TAQ) data for the time-weighted quoted bid–ask spread, trading volume, and the number of trades. We use Center for Research in Security Prices (CRSP) data for stock prices, market capitalization, and the volatility of stock returns. Variables are winsorized at the 1% level. We require stocks to be U.S.-listed common stocks (SHRCD 10 or 11) with a standard lot size of 100 shares and prices higher than $1 per share. Our main sample period is the year 2019, the most recent period for which we have one full year of data.

[Insert Table 2 about Here]

Panel A of Table 2 strongly rejects the null hypothesis that $\delta = 1$. Therefore, the percentage bid–ask spread depends strongly on the dollar lot size. The results reported in column (1) show that $\delta = 2.08$, which squares with our model's prediction of $\delta = 2$. The coefficients for volatility and trading volume are quantitatively close to 1. Our parsimonious three-factor model captures most of the cross-sectional variation in the bid–ask spread, with an R-squared as high as 0.83. Columns (2) – (5) show similar results prior to 2019, indicating the robustness of the Modified Square Rule.

In Panel B of Table 2 we report the results of comparing the three-factor model with two canonical benchmarks: Madhavan (2000) and Stoll (2000). Following their specifications, we normalize all dependent variables by price. The results reported in column (1) show that the R-squared of the three-factor model (0.84) is much higher than that in Madhavan (2000; see column 2, 0.61) and Stoll (2000; see column 3, 0.61), even though the three-factor model includes only a subset of variables included in previous benchmarks. This improvement in the goodness of fit is surprising, because adding more explanatory variables should, at a minimum, mechanically increase the R-squared. In columns (4)–(8) we then report results indicating two explanations of this surprising outperformance. First, our model provides a better functional form for each variable. Second, our model enables us to remove redundant variables.

Regarding the first point, Madhavan (2000) uses $price^{-1}$ to control for the relative tick size, while Stoll (2000) uses $log(price)$. Both specifications imply a monotonic relationship between the price and the percentage spread. In reality (as seen in Figure 1), the relationship between price and liquidity is U-shaped. Therefore, these two canonical benchmarks may misspecify the relationship between price and liquidity, at least in recent years. Another indicator of such misspecification is the coefficient estimate for the price. For example, the Madhavan (2000) specification shows that an increase in the price or a decrease in the relative tick size increases the percentage spread, despite overwhelming evidence derived from natural experiments that the percentage spread should increase with the tick size (Bessembinder 2003; Albuquerque, Song, and Yao, 2020). Stoll's (2000) specification shows that the price does not correlate with the percentage spread in our sample, although the economic reasoning in Stoll (2000) suggests that the price should matter. Our model provides an interpretation that applies to this puzzle: the price matters for liquidity; it just does not matter in a linear way.

Our model indicates that a better functional form in the regression is to subtract one tick from the bid–ask spread to control for the tick size and use log(price) to control for the lot size. To obtain the results reported in columns (4) and (5), we change the specifications in Madhavan (2000) and Stoll (2000) in only one respect: we replace the dependent variable in their regressions, the percentage spread (the bid–ask spread divided by the price), with the log of the percentage lot-driven spread ((bid–ask spread – 1 cent) divided by price). This one-cent change makes a huge difference. The R-squared in Madhavan's (2000) specification increases from 0.61 to 0.71 while the R-squared in Stoll's (2000) specification increases from 0.61 to 0.83. Also, the coefficients for prices become statistically more significant.

Regarding the second point, our model suggests that we can remove market cap in the regression. Almost all empirical tests of liquidity control for the market cap, reflecting the intuition that large-cap stocks should be more liquid (Stoll 2000; Madhavan 2000). The results reported in column (2) of Table 2 show that adding the market cap only marginally improves the explanatory power. Interestingly, Stoll (2000) documents a similar puzzle in his sample period: the market cap has very weak explanatory power for the percentage spread, and an increase in the market cap can increase the percentage bid–ask spread (Table 1, p. 1481). Our model provides the intuition that explains why the market cap has almost no additional explanatory power for the percentage spread. Madhavan (2000) and Stoll (2000) also control for the dollar volume, and our model suggests that the market cap becomes a redundant variable after we control for the dollar volume. Notice that we model market cap as v_t, and it affects liquidity only through its product with λ_I, the turnover rate. Our model predicts that a small-cap stock with high turnover is as liquid as a large-cap stock with low turnover if they have the same dollar volume. The results reported in column (6) show that adding the market cap to our three-factor model increases the R-squared by only 0.01. The results reported in column (7) show that the R-squared declines from 0.84 to 0.76 if we remove the dollar volume but keep the market cap. In summary, although the market cap appears to be a universal explanatory variable in most regressions, it does not directly affect the market maker's decision regarding the bid–ask spread, likely because the competitive market maker care more about the dollar trading volume that pays the bid–ask spread and less about the size of the firm *per se*.

5.1 A two-factor empirical model of nominal prices

Proposition 3 predicts that a firm's optimal nominal price is $p_t^* = \sqrt{\frac{\lambda_I \Delta v_t}{2\sigma\lambda_J L}}$. Taking the natural log on both sides, we obtain:

$$log(p_t^*) = \frac{1}{2}log(\lambda_I v_t) - \frac{1}{2}log(\sigma\lambda_J) + const. \qquad (12)$$

Rewriting (12) as a cross-sectional test gives us:

$$log(Price)_i = \frac{1}{2}log(Dollar\ Volume)_i - \frac{1}{2}log(Volatility)_i + \varepsilon_i. \qquad (14)$$

In Panel C of Table 2 we report he cross-sectional results for nominal prices. In column (1) we show that the two-factor model captures 61% of the cross-sectional variation in stock prices. An increase in volatility decreases the nominal price. This result is consistent with Baker, Greenwood, and Wurgler's (2009) finding on stock splits, according to which *"a somewhat unexpected result is the effect of volatility, which suggests that volatile firms have a greater, not lesser, propensity to manage prices downward."* Our model rationalizes their puzzle. An increase in volatility increases adverse selection risk for market makers, and firms with higher volatility should choose a lower price to reduce the dollar lot size. Another way to understand these results is that firms with higher volatility have a higher percentage spread and are less tick-constrained. Therefore, these stocks can achieve their optimal 2-tick spreads at lower nominal prices.

We also find that the nominal price increases with dollar volume. As stocks that trade in higher volumes tend to be larger stocks, we provide an interpretation for the observations in Baker, Greenwood, and Wurgler (2009) and Weld et al. (2009) that large stocks choose higher prices. An increase in the dollar volume reduces the percentage spread and also the tick-size constraints. Therefore, firms tend to choose higher prices to relieve tick-size constraints.

Weld et al. (2009) use industry fixed effects to explain nominal prices in the cross-section. We reconfirm their results, as reported in columns (2) and (3). Starting with a univariate regression with log volume and adding industry fixed effects increases the R-squared from 0.46 to 0.52. When we add volatility to the regression, though, the industry fixed effects increase the R-squared by only 0.01, as reported in column (4). Therefore, volatility subsumes most of the explanatory power of industry fixed effects. One rational interpretation for the industry clustering found in Weld et al. (2009) is that firms in the same industry may be subject to similar volatility.

In summary, we find that our model fits qualitatively with cross-sectional variations in nominal prices. The fit is less perfect than the fit for the bid–ask spread (0.61 vs. 0.84), and the coefficient on the estimate does not change one for one with model predictions. Interestingly, it is this imperfect fit that enables us to identify the impact of prices on the bid–ask spread. If all firms chose their prices following our model, log(price) would correlate almost perfectly with log(volatility) and log(dollar volume), leading to collinearity. There are two possible, albeit not mutually exclusive, interpretations of the less perfect fit of firms' behavior than of traders' behavior. First, our model overlooks other important drivers of the firm's choice but not of the traders' choice. Second, firms respond to market structure frictions to a lesser extent than traders do. Therefore, firms may end up with suboptimal nominal prices. We cannot rule out the first interpretation, but we find empirical evidence consistent with the second interpretation through stock splits.

6. Liquidity and Returns around Stock Splits

In this section, we use stock splits as a laboratory to test the implications of our model. In Section 6.1, we describe our data and sample. In Subsection 6.2, we show that our model matches changes in the percentage spread after splits. In Subsection 6.3, we show that most splits are correct because they tend to increase liquidity. In Subsection 6.4, we find that model-predicted changes in the percentage spread can explain the cross-sectional variation in announcement returns on splits.

6.1 Data, Sample, and Summary Statistics

Our sample includes all U.S. common stock–split announcements (CRSP event code 5523) from June 2003 through December 2019.[7] We require stocks to be U.S.-listed common stocks (the SHRCD is 10 or 11) and have pre- and post-split prices higher than $1 per share. We use CRSP data for stock-split ratios and announcement dates, split-adjusted stock returns, and market returns around declare dates as well as control variables. We use millisecond TAQ data to calculate the time-weighted quoted bid–ask spread and the quoted NBBO depth. To calculate cumulative abnormal returns (CARs), we obtain daily Fama-French factor returns and risk-free rates from Kenneth French's data library. We also require that the declaration date, the ex-date, and the split ratio be neither missing nor duplicated from CRSP. In addition, we use COMPUSTAT data to obtain annual reported numbers of shareholders and we aggregate 13-F filings to calculate the institutional

[7] The sample period begins in the month in which the millisecond TAQ data become available.

holdings of a stock one quarter before and after its split announcement. Variables are winsorized at the 1% level. Following Grinblatt, Masulis, and Titman (1984), we require that stock-split ratios be greater than or equal to 1.25 (5-for-4).[8] We end up with 1,183 stock splits.

In Table 3 we report the descriptive statistics. We have 912 unique stocks in our sample. The most common splits are 2-for-1 splits (645 times) and 1.5-for-1 splits (355 times), and the mean split ratio is 1.90. The average price before a split announcement is $57.97 and the average price after a split is $32.38. Also, the average number of trades increases from 2,718 trades per day to 4,447 per day, a 64% increase. Yet the dollar trading volume experiences almost no change ($44.25 million compared with $44.72 million). This supports our hypothesis that execution algorithms slice and dice their latent interest into smaller dollar size after stock splits. Also, institutional holdings increased slightly, from 57.90% to 58.04%, indicating that retail traders' holdings do not change dramatically. Therefore, changes in the compositions of retail/institutional holdings are unlikely to drive our results.

[Insert Table 3 about here]

6.2. Spread Changes around Stock Splits

In this subsection we show that changes in percentage spreads fit closely with the Modified Square Rule. We measure the bid–ask spread before splits as the average bid–ask spread 180 to 60 days before a split announcement day.[9] We define R_i as the predicted change in the percentage spread:

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[8] The CRSP does not record reverse-split announcement dates. Also, reverse-split announcements are usually mechanical and associated with bad news. For example, firms reverse-split to comply with an exchange's listing requirement of a $1.00 minimum bid price (Martell and Webb 2008).

[9] Stock trades around split announcements are volatile (Ohlson and Penman, 1985). Therefore, when measuring the bid–ask spread we exclude 60 days around the split window and consider the spread difference between the two relatively calm periods before the announcement and after the ex-date.

$$R_i = \frac{(s_i^{bef} - \Delta)/H_i^2 + \Delta}{p_i^{bef}/H_i} - \frac{s_i^{bef}}{p_i^{bef}}, \tag{15}$$

where $\frac{s_i^{bef}}{p_i^{bef}}$ is the percentage spread before the split and $\frac{(s_i^{bef} - \Delta)/H_i^2 + \Delta}{p_i^{bef}/H_i}$ is the percentage spread predicted by the Modified Square Rule (Corollary 2).

Next, we examine whether our predicted spread change R can explain actual spread changes after splits. We run the following regression:

$$\Delta \mathcal{S}_i = \beta \cdot R_i + Controls_i + Industry_i \times Year\ FE_t + \varepsilon_i. \tag{16}$$

The realized change in the percentage spread, $\Delta \mathcal{S}_i$, is the difference between the average percentage spread 180 to 60 days before announcement days and the average percentage spread 60 to 180 days after ex-dates. Following Weld et al. (2009), our control variables include market capitalization, price, volume, and turnover rates.

[Insert Table 4 about here]

Table 4 shows that a predicted 1 bp increase in the spread leads to a 0.97 bps realized increase in the spread, with *t*-statistics as high as 4.87. Therefore, the Modified Square Rule strongly predicts the percentage spread after splits.

6.3. Correct versus Incorrect Splits

After showing that the changes in realized percentage spreads exhibit an almost one-for-one match with the Modified Square Rule, we consider whether a firm's decision to split and its split ratio improve liquidity. We find that firms in general make the correct decisions.

Our model predicts that a split is correct if it moves the bid–ask spread closer to the two-tick optimum. Mathematically, a split is correct if $R_i < 0$ and a split is incorrect if $R_i > 0$, and R_i reaches its minimum if the split ratio leads to the optimal two-tick spread. We find that 1,077 splits are "correct" and 106 splits are "incorrect."

Among the 106 incorrect splits, 73 should have split, because their bid–ask spreads are higher than the 2-tick optimum. However, they choose split ratios that are so aggressive that their new bid–ask spreads are further away from the 2-tick optimal. We find that R_i, on average, decreases by 15.22 bps in our sample, providing additional evidence that splits are in general correct.

6.4. Cumulative Abnormal Returns around Announcements

Liquidity affects asset value (Amihud and Mendelson, 1986), and Figure 2 presents preliminary evidence that our model-predicted liquidity change affects returns on split announcements: firms in the group with correct splits realize an average announcement CAR of 2.88%, whereas those in the group with incorrect splits obtain an average announcement return of only 1.26%.[10]

[Insert Figure 2 about Here]

Insofar as splits are good news in general, both groups enjoy positive returns, but the 1.62% difference indicates that predicted liquidity changes may contribute to the difference in returns. To test this hypothesis, we run the following regression:

$$CAR_{i,[T-1,T+1]} = \theta \cdot R_i + Controls_i + Industry_i \times Year\ FE_t + \varepsilon_i. \quad (24)$$

Following Weld et al. (2009), our control variables include market capitalization, price, volume, and turnover rates. We also control for industry-year fixed effects to absorb any industry and time-specific shocks, where each industry is defined by reference to the first two digits of the NAICS classification. We also control for institutional holding changes and the number of investor changes [following Amihud, Mendelson, and Uno (1999) and Dyl and Elliott (2006)] to exclude the impact of investor base changes.[11]

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[10] Following Grinblatt, Masulis, and Titman (1984), we consider the window of announcement abnormal returns as dates -1, 0, and 1.

[11] These variables are missing for more than half of the firms, so we do not require them in the baseline test. As the results reported in column (5)–(7) of Table 5 show, our results are robust to adding these controls for the reduced sample.

[Insert Table 5 about here]

The results reported in Table 5 show that our predicted spread change is significantly negatively associated with split-announcement abnormal returns. Column (1) show that a 1 bps predicted increase in the percentage spread is associated with -5.49 bps in announcement returns.[12] After adding control variables, the results reported in column (3) show that a 1 bps predicted increase in the percentage spread is associated with -6.25 bps in announcement returns. As the mean of R_i is -15.22 bps, correct split ratios contribute $-15.22 \times -6.25 = 95$ bps to the overall average split-announcement abnormal return of 272 bps. Therefore, a reduction in market microstructure friction provides a partial explanation of why stock split, a seemingly cosmetic change, lead to positive returns.

The Table 5 results show that the explanatory power of the tick-and-lot channel is orthogonal to two existing interpretations of splits and announcement returns from splits. Brennan and Copeland (1988) propose that firms use splits to convey positive signals about firm fundamentals, and the cost of such signals is reduced liquidity. Brennan and Copeland (1988) predict, therefore, that a larger reduction in liquidity should be a stronger signal and is associated with higher returns. We find, however, that splits improve liquidity and a greater improvement in liquidity leads to a higher return. Both patterns are inconsistent with the signaling channel. Lamoureux and Poon (1987) and Maloney and Mulherin (1992) propose that firms use stock splits to attract retail traders, and an increase in uninformed traders increases volume and liquidity. We found in Table 3 that institutional holdings increased slightly after stock splits. The results reported in column (4) of Table 5 show that the change in retail holdings, proxied by the number of shareholders and

[12] The economic magnitude is similar to that reported in Albuquerque, Song, and Yao (2020). Using a controlled experiment, they find that a 43.5 to 48.2 bps increase in the bid–ask spread led to a 175 to 320 bps drop in asset values.

institutional holdings, does not affect announcement returns.

7. Economic gains from better management of nominal prices

In Sections 5 and 6, we provide evidence of the value of managing nominal prices, and we also show that firms on average move their nominal prices in the right direction during stock splits. However, the existence of pairs of stocks with similar fundamentals but dramatically different nominal prices suggests that one stock in the pair should have an incorrect price. For example, for 2019, Amazon brought an average price of around $1,800 and Apple brought a price of $200. Ford brought a price of $9 and GM brought a price of $38.

The Two-Tick Rule and the Modified Square Rule provide predictions as to which stock in each pair has an incorrect price, and the empirical evidence is consistent with such predictions. Amazon's price is too high, because its bid–ask spread (60 cents) is further away from the 2-cent optimal than Apple's bid–ask spread (1.81 cents). Indeed, Amazon's percentage spread is 3.3 bps, whereas Apple's percentage spread is only 0.9 bps. The Modified Square Rule shows that differences in nominal prices can almost fully explain the fourfold dramatic differences in transaction costs. If Amazon implements a 9-for-1 split, it will have a nominal price that is similar to Apple's, and its nominal spread, under the Modified Square Rule, would be $\frac{60-1}{9^2}+1=1.73$ cents. In fact, the optimal split ratio for Amazon is $\sqrt{60-1}=7.68$-for-1, with which Amazon would achieve the optimal two-tick bid–ask spread at $235, and the percentage spread further reduces to 0.85 basis points. The reduction in transaction costs would save Amazon investors $232 million per year. Amazon's market cap would increase by $1.35 billion based on our estimated elasticity of firm value to the percentage spread. We then find that the Modified Square Rule can explain why Ford's percentage spread (11 bps) is four times greater than General Motors' (2.7 bps). This is because Ford's price is too low. If Ford were to implement a 1-for-4 reverse split, its percentage spread would be similar to General Motors'. Ford investors would save $36 million per year if the company were to choose a price that is similar to GM's.

Figure 3 formalizes the intuition presented in the previous two anecdotes. We extend Figure 1 by adding three dashed lines that show the optimal percentage spread if all firms in the basket choose the optimal prices predicted by our model, that is, the nominal price that generates a two-tick bid–ask spread. The horizontal axis presents the current price of the stock and the vertical axis presents the percentage spread. A larger vertical gap between the solid and dashed lines implies a greater economic gain.

[Insert Figure 3 about here]

For large stocks, the biggest winners would be those with low prices such as General Electric, Ford, Bank of America, SiriusXM, and Sprint. Almost no firms voluntarily split in this segment of the market: most of these stocks experienced very large price slides before the sample period and have not fully recovered. Their low prices led to binding tick sizes, and they almost always trade at a one-cent spread, which results in a very large percentage spread. We conjecture that such firms do not reverse-split because reverse splits are usually regarded as negative "signals," and these firms would rather wait for (possible) price recovery to ease the binding tick size. We encourage these firms to ignore the negative connotations of reverse splits and escape the tick-binding restriction. A good example to follow is Citigroup, which announced a 1-for-10 reverse split on March 21, 2011, when its stock was trading at around $4 as a result of its 90%+ loss in market cap. In untabulated results, we find an increase in liquidity for Citigroup after the reverse splits.

On the other side, a small firm should not choose a high price. Such a firm might choose a high price because people often consider a high-priced stock a prestigious stock (Weld et al. 2009). The cost of maintaining a minimum lot of 100 shares in liquidity is, however, very high for a small stock with a price higher than \$70, leading to a large percentage spread for these stocks.

We find that the median optimal price for large stocks (NYSE deciles) is \$53.89, whereas the median small stock can sustain an optimal price of only \$4.05. The results presented in Figure 3 suggest that a small-cap stock should not choose a high price, whereas a large stock should not choose a low price.

Finally, we estimate the potential liquidity improvement that can be obtained with optimal pricing. After adopting optimal pricing, the mean spreads will be reduced from 29.87 bps to 18.89 bps, a 37% reduction. For small-cap stocks, the spread will decrease from 87.67 bps to 49.43 bps. For medium-cap stocks, the spread will decrease from 14.35 bps to 9.88 bps. The spread for large-cap stocks will decrease from 5.26 bps to 3.71 bps. Because the sensitivity of prices to liquidity changes that we report in Table 4 is 6.25, we expect the value of the median U.S. stock to increase by (29.87-18.89) $\times$ 6.25 = 69 bps after adopting optimal pricing. Small stocks tend to be the biggest winners by achieving optimal nominal prices, and their value will increase by 239 bps if they choose their best nominal prices, but the median large-cap stocks can also increase their value by 9.69 bps. Summing up the potential gains for each stock, the total benefit of adopting optimal pricing is estimated to be \$54.9 billion. The top benefiting firms are Alphabet (\$1.89 billion), Amazon (\$1.35 billion), and Bank of America (\$549 million).

8. Conclusion

Economic models often incorporate an implicit but important assumption—continuous pricing and continuous quantities. In this paper, we offer the first model where both prices and quantities are discrete, and we show that these two seemingly small frictions can lead to significant economic impacts. Firms should set their nominal prices such that the friction cause by discrete pricing is equal to the friction caused by discrete lots, and all firms achieve their optimal price when their nominal bid–ask spreads equal two ticks. Overall, we find that most stock splits are correct, and the resultant liquidity improvement contributes 95 bps points to the average split-announcement return of 272 bps. We estimate that the median U.S. stock value would increase by 69 bps if all firms were to move to their optimal prices and total market value would increase by $54.9 billion.

Our paper focuses on pricing choices in firms, but we also provide a consistent interpretation of the improvement in liquidity after trading becomes automated. Automated trading allows market makers and liquidity demanders to slice their orders into a series of minimum lots. Once market makers can quote a minimum lot and immediately refill it once it is consumed, their exposure to adverse selection declines and they can quote a tighter spread at reduced depth. Our interpretation is the first that matches four stylized facts simultaneously: a reduction in the percentage bid–ask spread, a reduction in depth towards one lot, the dominance of trades of exactly 100 shares, and the rise of algorithmic traders who are slower than HFTs.

Our parsimonious model explains 83% of the cross-sectional variation in the bid–ask spread. One possible driver of this goodness-of-fit success is that our model may capture the main drivers of the bid–ask spread. Interestingly, our model uses a subset of the parameters modeled in Madhavan (2000) and Stoll (2000). Therefore, we provide the theoretical foundations for their empirical model and propose a more powerful functional form. We encourage researchers to consider our empirical specification when they search for new explanatory variables related to liquidity, either in the cross-section or when they evaluate the impact of policy shocks. First, the dependent variable should be the bid–ask spread minus one tick, because the one tick comes simply from tick-size constraints. The log(price) should then be added as a control variable to control for lot-driven spreads.

Our paper offers two policy implications. First, we discourage the initiative to increase the tick size from one cent to five cents because it reduces liquidity, and we encourage the initiative to decrease the lot size because it improves liquidity. Second, we find that the move to a proportional tick-and-lot system reduces liquidity, if regulators choose the tick and lot size of any existing stock under the uniform system as the benchmark. The economic intuition behind this surprising result is that the uniform system is actually more flexible than the proportional system. The first best in our model is perfectly continuous pricing and quantities. A uniform system offers one degree of freedom because it allows firms to pursue continuous pricing more fully at the cost of more discrete quantities, and vice versa. A proportional system tends to reduce liquidity because it further removes the only degree of freedom for firms by imposing the same level of discreteness in pricing and quantities for all firms despite their heterogeneity.

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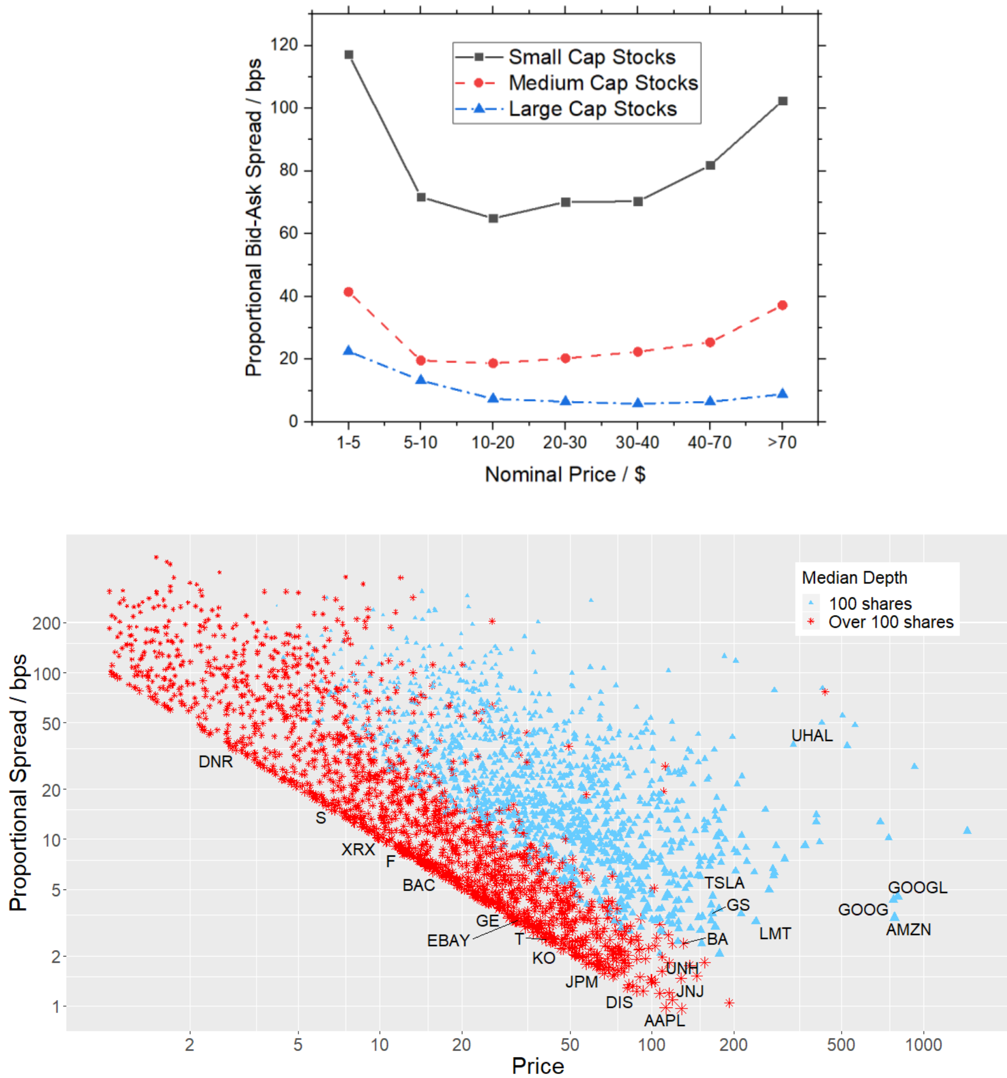

Figure 1 (U-Shaped relationship between liquidity and prices). These figures show the relationship between the percentage spreads and nominal prices. Our sample includes all U.S.-listed common stocks that have a 1-cent tick size, a 100-share lot size, and at least a $1 nominal price. For Panel A, we take the average spread across price baskets and stratify by market caps. The square, circle, and triangle lines are small-, medium-, and large-cap stocks, respectively. Price baskets are selected such that each basket contains a similar number of stocks. In Panel B, we plot each firm as a triangle or asterisk, where larger shapes represent larger market-cap firms. Blue triangles represent stocks with 100 shares of the median national best bid/offer (NBBO) depth while red asterisks represent stocks with median NBBOs of more than 100 shares. The bottom-left boundary represents the 1-cent tick size constraint.

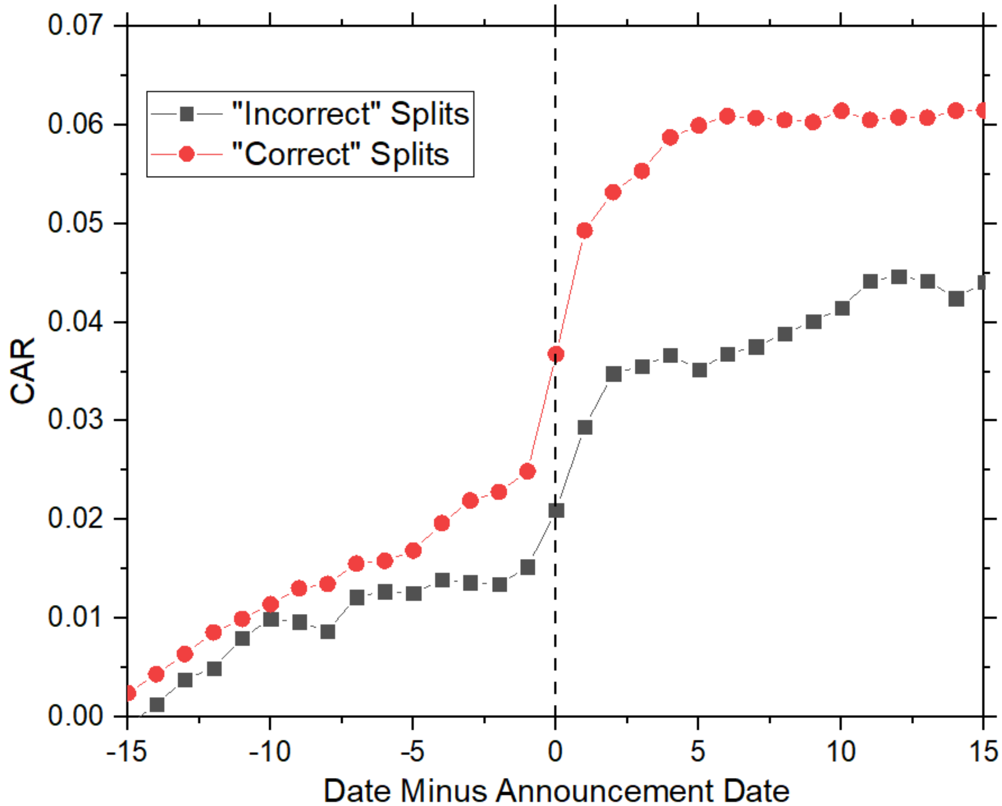

Figure 2 (Split Announcement Returns). This figure shows the cumulative abnormal returns (CARs) around split-announcement dates. Our sample includes all U.S.-listed common stock splits beginning in September 2003. We require the firm to have at least a \$1 nominal price before and after a split. We categorize stocks into two types based on Proposition 3. A split is "correct" if Proposition 3 predicts a decrease in the percentage spread and "incorrect" if Proposition 3 predicts an increase in the percentage spread.

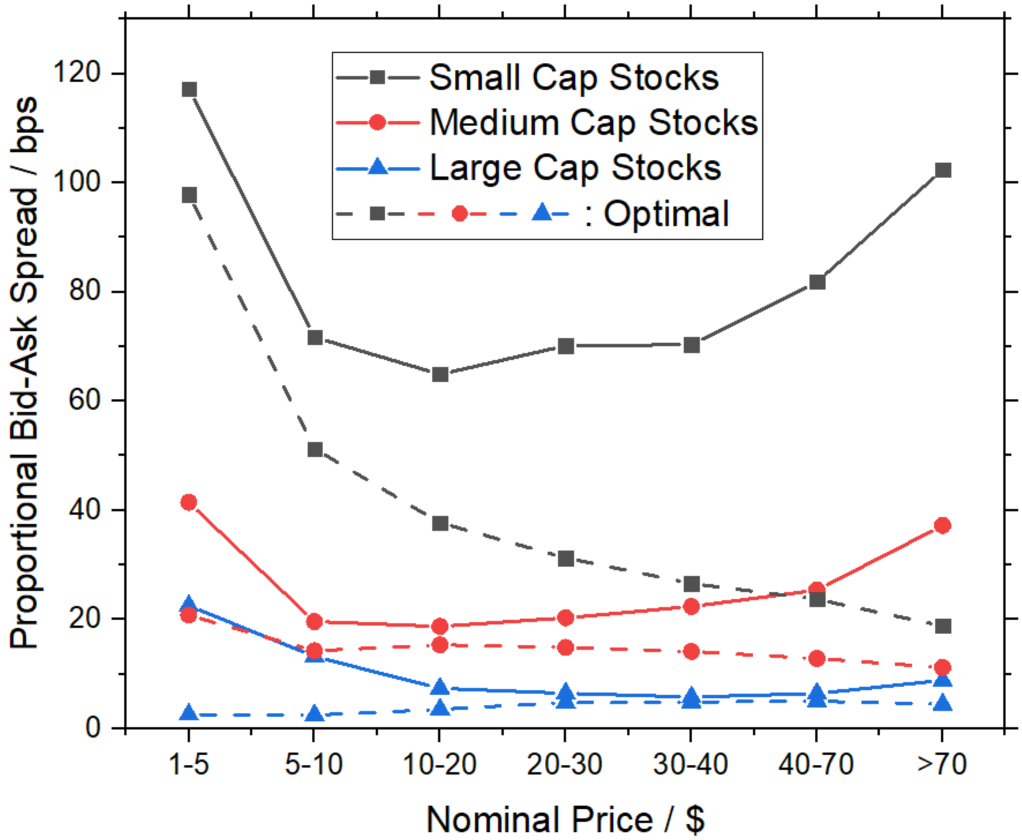

Figure 3 (Economic gain from adopting optimal nominal prices). This figure shows the relationship between average percentage spreads and nominal prices. Our sample includes all U.S.-listed common stocks that have a 1-cent tick size, a 100-share lot size, and at least a $1 nominal price. The square, circle, and triangle lines are small-, medium-, and large-cap stocks, respectively. Price baskets are selected such that each basket contains a similar number of stocks. The solid lines are observed percentage spreads for stocks within the same price–size basket, and dashed lines are theoretical possible minimum spreads of the stocks in the same baskets.

Table 1
Optimal Price and Percentage spread with Fixed and Proportional Tick/Lot Sizes

Tick Size \ Lot Size	Fixed $L = \frac{1}{\ell}$	Proportional $\mathbb{L}(P_t, v_t) = k^L/(P_t v_t)$
Fixed Δ	$p_t^* = \sqrt{\frac{\lambda_I \Delta v_t}{2\sigma\lambda_J L}}, \mathcal{S}^* = \frac{2\Delta}{p_t^*}$	$p_t^* \to \infty, \mathcal{S}^* = \frac{2\sigma\lambda_J}{\lambda_I v_t} k^L$
Proportional $\mathbb{D}(P_t, v_t) = k^\Delta P_t v_t$	$p_t^* \to 0, \mathcal{S}^* = k^\Delta$	$\forall p_t, \mathcal{S}^* \equiv k^\Delta + \frac{2\sigma\lambda_J}{\lambda_I v_t} k^L$

In this table we summarize the firm's optimal choices regarding price p_t^* and the corresponding minimum percentage spread $\mathcal{S}^*$ under various tick- and lot-size systems. Continuous pricing is a special case for the first row, where $\Delta = 0$, and continuous quantity is a special case for the first column, where $L = 0$. Proportional tick and lot sizes are summarized in the second row and column, respectively. Reads: With fixed tick and lot sizes, firms choose the optimal nominal price p_t^*. Firms split (reverse split) to the extreme if the tick (lot) size is proportional to the nominal price. If both the tick and lot sizes are proportional, the firm's liquidity is irrelevant to the nominal price.

Table 2
Lot-driven Spread and the Square Rule
Panel A: Three-Factor Model on Liquidity

	(1)	(2)	(3)	(4)	(5)
Dependent Variable	$Log(s_t^L) = Log(s_t^{tot} - \Delta)$				
Sample Period	2019	2018	2017	2016	2015
Log(Price$_t$)	**2.08*** **	**2.12*** **	**2.06*** **	**2.08*** **	**2.06*** **
	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)
Log(Volatility$_t$)	**1.19*** **	**1.16*** **	**1.07*** **	**1.17*** **	**1.14*** **
	(0.03)	(0.03)	(0.03)	(0.03)	(0.03)
Log(Volume$_t$)	**-0.82*** **	**-0.81*** **	**-0.79*** **	**-0.83*** **	**-0.81*** **
	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
Obs.	3652	3736	3711	3713	3850
R^2	0.8389	0.8003	0.7704	0.8095	0.8298
Adj. R^2	0.8387	0.8001	0.7702	0.8093	0.8296

Panel B: Specification Horseraces for the Three-Factor Model

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Dependent Variable	$Log(\mathcal{S}_t^L)$	$\frac{\mathcal{S}_t^{tot}}{2}$(bps)	$\frac{\mathcal{S}_t^{tot}}{2}$(bps)	$Log(\mathcal{S}_t^L)$	$Log(\mathcal{S}_t^L)$	$Log(\mathcal{S}_t^L)$	$Log(\mathcal{S}_t^L)$	$Log(\mathcal{S}_t^L)$
Sample Period	2019	2019	2019	2019	2019	2019	2019	2019
Log(Price$_t$)	**1.08*** **		-2.31		**0.31*** **	**1.19*** **	**1.24*** **	**1.19*** **
	(0.02)		(1.63)		(0.03)	(0.03)	(0.03)	(0.03)
Log(Volatility$_t$)	**1.19*** **					**0.99*** **	**0.39*** **	**1.01*** **
	(0.03)					(0.04)	(0.04)	(0.03)
Log(Volume$_t$)	**-0.82*** **	**-32.84*** **	**-13.91*** **	**-0.65*** **	**-0.90*** **	**-0.64*** **		
	(0.01)	(0.73)	(2.12)	(0.02)	(0.05)	(0.03)		
Log(MKTCAP$_t$)		**24.70*** **	**15.35*** **	-0.05	**-0.44*** **	**-1.35*** **	**-1.25*** **	**-0.98*** **
		(1.11)	(1.11)	(0.03)	(0.02)	(0.02)	(0.02)	(0.02)
Log(Turnover$_t$)								**-0.67*** **
								(0.03)
Log(#Trades$_t$)			**-13.62*** **		**0.29*** **			
			(2.22)		(0.05)			
*Volatility$_t$ * 10^2*		**12.18*** **		**0.19*** **				
		(0.47)		(0.02)				
*Variance$_t$ * 10^4*			**0.56*** **		**0.02*** **			
			(0.03)		(0.00)			
1/(Price$_t$)		**-15.65*** **		**-3.92*** **				
		(5.79)		(0.15)				
Obs.	3652	3652	3652	3652	3652	3652	3652	3652
R^2	0.8389	0.6088	0.6137	0.7053	0.8331	0.8497	0.7577	0.8536
Adj. R^2	0.8387	0.6084	0.6131	0.7050	0.8329	0.8496	0.7575	0.8535

Panel C: Two-Factor Model on Nominal Prices

	(1)	(2)	(3)	(4)
Dependent Variable		$Log(Price)_t$		
Log(Volatility$_t$)	**-0.85*** **			**-0.79*** **
	(0.02)			(0.03)
Log(Volume$_t$)	**0.26*** **	**0.32*** **	**0.32*** **	**0.27*** **
	(0.01)	(0.02)	(0.03)	(0.01)
Industry FE	N	N	Y	Y
Obs.	3652	3652	3652	3652
Adj. R^2	0.6087	0.4615	0.5196	0.6198

In this table we report the results of testing the modified square rule on the cross-section of U.S. common stocks. In Panel A we report the results of regressing the log of lot-driven nominal spreads on the log of nominal prices, controlling for log(Volatility) and log(Volume). We take a snapshot of the most recent five years of U.S. listed common stocks as our sample, and we take the annual average of the data. We require the stocks to have the standard 100-share lot size, a price higher than $1 during the entire year, and at least 20 observations within the year. In Panel B we use the results to compare the modified square rule with alternative specifications. In column (1) we report the results derived with our baseline model, while for columns (2) and (3) we incorporate the specifications of Madhavan (2000) and Stoll (2000), respectively. For columns (4) and (5) we use our model's dependent variable and Madhavan's (2000) and Stoll's (2000) independent variables. For columns (6)–(8) we estimate our model with alternative control variables. In Panel C we report the cross-sectional determinants of nominal prices. Coefficient estimates are shown in bold and standard errors are shown in parentheses. ***, **, and * denote significance at the 1%, 5%, and 10% levels, respectively.

Table 3
Summary Statistics

	Mean	Min	Q1	Median	Q3	Max	Std.Dev	N
Split Factor	1.9044	1.2500	1.5000	2.0000	2.0000	50.0000	1.5207	1183
Pre-split price ($)	57.9723	3.4940	31.6275	47.0800	70.0650	3311.0000	102.5506	1183
Post-split price ($)	32.3829	2.1500	21.3675	29.8900	39.9750	107.0950	15.5480	1183
Market cap ($MM)	6.1464	0.0087	0.4174	1.5059	4.2878	564.9798	21.6686	1183
Ex-ante spread (cents)	15.4359	1.0426	4.1980	6.9838	16.3665	294.5383	24.1887	1183
Ex-post spread (cents)	8.7920	1.0259	2.7791	4.2592	8.8398	98.4850	11.8502	1183
Predicted Spread Change (cents)	-9.7442	-290.6029	-9.1517	-3.9425	-2.1255	-0.0237	19.2978	1183
Predicted Spread Change (bps)	-15.2158	-497.0664	-12.7405	-3.1512	-0.9849	7.7606	34.9463	1183
Announcement CAR (%)	2.7337	-28.6363	-0.1079	1.7536	4.3253	68.4449	5.8950	1183
Ex-date CAR (%)	0.3153	-38.1811	-1.9452	0.0464	1.9744	157.0690	6.5191	1183
Ex-ante Dollar Volume ($MM/day)	44.2560	0.0020	1.1757	8.9945	34.3150	5289.9349	178.3195	1183
Ex-post Dollar Volume ($MM/day)	44.7224	0.0040	1.5924	10.2766	36.5166	4776.0898	162.0932	1183
Pre-split number of trades (thousands)	2.7176	0.0030	0.2834	0.9970	2.7015	101.5315	5.9936	1128
Post-split number of trades (thousands)	4.4467	0.0020	0.4723	1.5330	4.1675	215.3385	10.0324	1150
Pre-split institutional holding (%)	57.9026	0.0000	32.1769	66.0525	84.4367	99.1116	30.9838	942
Post-split institutional holding (%)	58.0483	0.0000	32.8861	66.0563	84.1892	97.8309	30.1347	924
Pre-split number of holders (thousands)	14.9088	0.0010	0.4500	1.9500	7.1820	1234.0000	65.8571	753
Post-split number of holders (thousands)	16.4495	0.0010	0.4410	2.0445	7.7995	1426.0000	71.5956	758
Log(number of holders change ratio)	0.0249	-7.2497	-0.0579	-0.0060	0.1014	4.5505	0.6517	751

In this table we report the summary statistics for our stock-split sample for September 2003—December 2019. Institutional holdings are taken from 13-F filings for the quarters immediately before and after a split announcement date. The announcement and ex-date CARs are cumulated announcement returns during dates [T-1, T+1], following Grinblatt, Masulis, and Titman (1984). Shareholder numbers are taken from the years immediately before and after stock-split announcements, and the logs of the changes are reported following Amihud, Mendelson, and Uno (1999). Other pre-split variables are measured in the 180-day-to-60-day window before split announcement days, and post-split variables are measured in the 60-day-to-180-day window after split implementation days.

Table 4
Predictions of Changes in Bid–ask Spreads

Dependent Variable	Realized ΔS_i (bps)
	1
R_i *(bps)*	**0.97*****
	(0.20)
Controls	Y
Industry-Year FE	Y
Obs.	1183
Adj. R^2	0.357

In this table we report the results obtained from regressing realized changes in the percentage spread on predicted spread changes and announced split ratios with various controls. R_i is the model-predicted change in the percentage spread (in bps). The split ratio comes from the CRSP item FACSHR. Following Weld et al. (2009), our control variables include log(market cap), price, log(volume), and turnover rates. We also control for industry-year fixed effects to absorb any industry-year-specific shocks, where each industry is defined by reference to the first 2 digits of the NAICS classification. Coefficient estimates are shown in bold and standard errors are shown in parentheses. Standard errors are adjusted for both heteroscedasticities and within correlations clustered by firm. ***, **, and * denote significance at the 1%, 5%, and 10% levels, respectively.

Table 5
Predicted Spread Changes and Abnormal Returns on Announcements

Dependent Variable	$CAR_{i,[T-1,\,T+1]}$ (bps)			
	1	2	3	4
R_i *(bps)*	**-5.49*** **	**-4.73** **	**-6.25*** **	**-6.97** **
	(1.48)	(1.90)	(2.23)	(3.35)
Split Ratio$_i$		0.14	0.21	0.50
		(0.12)	(0.11)	(0.50)
$Log(MktCap_{i,t-1})$		**5.67*** **	**5.51** **	**8.76*** **
		(1.94)	(2.25)	(2.61)
$Log(Price_{i,t-1})$		**-6.30*** **	**-6.47*** **	**-9.59*** **
		(1.98)	(2.29)	(2.55)
$Turnover_{i,t-1}$		**6.92*** **	**6.57*** **	**10.27*** **
		(1.95)	(2.27)	(2.58)
$Log(Volume_{i,t-1})$		**-6.18*** **	**-5.81*** **	**-9.14*** **
		(1.92)	(2.23)	(2.54)
$Log(\frac{InstHldg_{t+1Q}}{InstHldg_{t-1Q}})$				6.29
				(3.86)
$Log(\frac{TOTSH_{t+1yr}}{TOTSH_{t-1yr}})$				-0.15
				(0.29)
Industry-Year FE	N	N	Y	N
Obs.	1183	1183	1183	607
Adj. R^2	0.067	0.138	0.169	0.239

In this table we report the results of regressing split-announcement CARs on predicted spread changes and announced split ratios with various sets of subsamples and controls. R is the model-predicted change in the percentage spread. The split ratio comes from the CRSP item FACSHR. Following Weld et al. (2009), for column (2) we control for log(market cap), price, log(volume), and turnover rates before the splits. Industry-year fixed effects are added to obtain the results reported in column (3) to absorb any industry-year-specific shocks, where each industry is defined by reference to the first 2 digits of the NAICS classification. For column (4) we also control for changes in institutional holdings and shareholders, following Dyl and Elliott (2006) and Amihud, Mendelson, and Uno (1999). Institutional holdings are aggregated from quarterly 13-F filings before and after split announcements, and the numbers of shareholders are obtained from the COMPUSTAT annual item CSHR. Coefficient estimates are shown in bold and standard errors are shown in parentheses. Standard errors are adjusted for both heteroscedasticities and within correlations clustered by firm. ***, **, and * denote significance at the 1%, 5%, and 10% levels, respectively.

Appendix A. Proofs of Theoretical Results

Proof of Proposition 1 and Corollary 1

Uninformed traders can choose their order size distribution $f(q)$, where $q \in \{1,2,\dots,\ell\}$ in round lots. $F(q)$ is the cumulative distribution function. We aim to show that $F(1) = 1$ is necessary for minimizing the transaction cost.

The competitive market maker quotes a liquidity schedule with ℓ layers. For the q^{th} lot, the market maker can trade only with uninformed traders' liquidity demanding orders that are larger or equal to q round lots. Conditional an uninformed order arrival, the execution probability for the q^{th} share is $(1 - F(q-1))$, which is a decreasing function of q and the decrease is strict when $f(q) > 0$. However, the probability to be adversely selected is independent of q, because the informed trader would adversely select all lots in the book once she arrives. Therefore, the break-even spread for the q^{th} share satisfies $\mathcal{S}_t^1 \le \mathcal{S}_t^2 \le \cdots \mathcal{S}_t^q \le \cdots \le \mathcal{S}_t^\ell$, where the q^{th} inequality is strict when $f(q) > 0$. Two subcases arise:

1. If $f(1) \neq 0$, we have $\mathcal{S}_t^1 < \mathcal{S}_t^2$. Therefore, the BBO only contains one lot. Then, uninformed traders achieve their best execution only when $f(1) = F(1) = 1$ because it is suboptimal to submit larger orders that walk up in the book.
2. If $f(1) = 0$, the market maker quotes more than one round lot at the BBO. This spread $\mathcal{S}_t^1 = \mathcal{S}_t^2$, however, is strictly worse than $\mathcal{S}_t^*$. It is because the arrival rate of uninformed traders in dollars is fixed and market maker suffers higher adverse selection risk than sustaining only one round lot of bids and asks.

Therefore, $F(1) = 1$ is the unique optimal execution strategy. Now we calculate the equilibrium percentage spread $\mathcal{S}_t^L$. Uninformed traders demand $\lambda_I v_t$ dollars of liquidity per unit time. Therefore, the per-unit-time profit of the market maker equals the per-unit-time adverse selection loss:

$$\lambda_I v_t \cdot \frac{\mathcal{S}_t^L}{2} = \lambda_J \cdot p_t L(\sigma - \frac{\mathcal{S}_t^L}{2}). \tag{A.1}$$

$$\mathcal{S}_t^L = \frac{2\sigma\lambda_J p_t L}{\lambda_I v_t + \lambda_J p_t L}. \tag{A.2}$$

Recall that $V := \lambda_I v_t + \lambda_J p_t L$ is the total dollar volume per unit time, we directly have Corollary 1 that the dollar bid-ask spread $s_t^L := \mathcal{S}_t^L p_t = \frac{2\sigma\lambda_J p_t^2 L}{V}$. ■

Proof of Proposition 2

The quoted bid-ask spread at $B_t = \left\lfloor \frac{p_t - s_t^L/2}{\Delta} \right\rfloor \Delta$ and $A_t = \left\lceil \frac{p_t + s_t^L/2}{\Delta} \right\rceil \Delta$ is competitive because any quotes improving the bid and ask prices by one tick would lose money. In this proof, we calculate the average widening effect in two steps. First, we show that under our Poisson jump process, p_t converges to a lognormal distribution, and the residual $\left\{\frac{p_t}{\Delta}\right\}$ tends to be uniformly distributed with in the tick. Second, we show that the uniform distribution lead to average widening effect of Δ, so the tick-constrained spread is one tick higher than the continuous case.

First, observe the process that v_0 jumps up or down by $\sigma\%$ following a Poisson process with intensity λ_J. Then, we have

$$v_t = v_0 \cdot (1+\sigma)^u \cdot (1-\sigma)^d, \tag{A.3}$$

where $u \sim Poisson(\frac{\lambda_J t}{2})$ and $d \sim Poisson(\frac{\lambda_J t}{2})$. Take log on both sides, we have

$$log(v_t) = log(v_0) + u \cdot log(1+\sigma) + d \cdot log(1-\sigma). \tag{A.4}$$

When the jump has happened for a sufficient number of times, i.e., $t \gg \frac{1}{\lambda_J}$, we apply the central limit theorem on (A.4) and $log(v_t)$ converges in distribution to a normal distribution with mean $\mu(t) = log(v_0) + \frac{\lambda_J t}{2} \cdot log(1+\sigma) + \frac{\lambda_J t}{2} \cdot log(1 -$

σ) and variance $\Phi(\mathrm{t}) = (\frac{\lambda_J t}{2} log(1+\sigma))^2 + (\frac{\lambda_J t}{2} log(1-\sigma))^2$. Then, v_t follows the lognormal distribution $\mathcal{LN}(\mu(t), \Phi(\mathrm{t}))$, and $p_t = \sqrt{\frac{\lambda_I \Delta v_t}{2\sigma\lambda_J}}$ follows the lognormal distribution $\mathcal{LN}(\frac{\mu(t)}{2}\sqrt{\frac{\lambda_I \Delta}{2\sigma\lambda_J}}, \frac{\Phi(\mathrm{t})}{4}\frac{\lambda_I \Delta}{2\sigma\lambda_J})$.

Next, we estimate the maximum value range of the probability distribution function within a tick. Let $g(p)$ be the pdf of the lognormal distribution. We compare $g(p+\frac{\Delta}{2})$ and $g(p-\frac{\Delta}{2})$ and show that for any $p \gg \Delta$, the relative difference $\left|\frac{g(p+\frac{\Delta}{2})-g(p-\frac{\Delta}{2})}{g(p)}\right|$ is as small as in the order of $\frac{\Delta}{p}$. With this estimation, the residual of p within a tick is almost uniformly distributed.

Since $p \gg \Delta$, we have $g(p+\frac{\Delta}{2}) - g(p-\frac{\Delta}{2}) \approx \Delta g'(p)$, and $\left|\frac{g(p+\frac{\Delta}{2})-g(p-\frac{\Delta}{2})}{g(p)}\right| \approx \left|\frac{\Delta g\prime(p)}{g(p)}\right|$. Inserting the pdf of the lognormal distribution $\mathcal{LN}(\frac{\mu(t)}{2}\sqrt{\frac{\lambda_I \Delta}{2\sigma\lambda_J}}, \frac{\Phi(\mathrm{t})}{4}\frac{\lambda_I \Delta}{2\sigma\lambda_J})$ into $f(p)$, we have:

$$\left|\frac{\Delta f\prime(p)}{f(p)}\right| = \frac{\Delta}{p}(1 + \frac{log(p) - \frac{\mu(t)}{2}}{\frac{\Phi(\mathrm{t})}{4}\frac{\lambda_I \Delta}{2\sigma\lambda_J}}). \tag{A.5}$$

When $t \to \infty$, Φ goes to infinity at the order of t^2, and $\frac{log(p) - \frac{\mu(t)}{2}}{\frac{\Phi(\mathrm{t})}{4}\frac{\lambda_I \Delta}{2\sigma\lambda_J}}$ becomes negligible. Thus, for any $p \gg \Delta$, the relative difference $\left|\frac{g(p+\frac{\Delta}{2})-g(p-\frac{\Delta}{2})}{g(p)}\right|$ is on the order of $\frac{\Delta}{p}$. The difference is the largest when $\frac{p}{\Delta}$ is the smallest (i.e., when $p =$

\$1.005 and $f(\$1.00)/f(\$1.01) \approx 10^{-2}$ if $\Delta = \$0.01$). For a median \$35 stock, the maximum range is even smaller at $\frac{1}{3500}$ and mostly negligible.[13]

The last step is to show that the widening effect is one tick with a uniformly distributed $\left\{\frac{p_t}{\Delta}\right\}$. Intuitively, if the midpoint price p_t is uniformly distributed in the sub-tick granularity, the bid and ask prices $p_t \pm \frac{s_t^L}{2}$ are both uniformly distributed in the sub-tick granularity for any s_t^L. The problem is symmetric. Without loss of generality, we consider the bid side $= \left\lfloor\frac{p_t - s_t^L/2}{\Delta}\right\rfloor \Delta$, where s_t^L can be any positive number and $\left\{\frac{p_t}{\Delta}\right\} \sim U[0,1)$, where $\{x\}$ is the fractional part of x. We have:

$$B_t = \left\lfloor\frac{p_t - s_t^L/2}{\Delta}\right\rfloor \Delta = \left(\frac{p_t - \frac{s_t^L}{2}}{\Delta} - \left\{\frac{p_t - \frac{s_t^L}{2}}{\Delta}\right\}\right)\Delta = p_t - \frac{s_t^L}{2} - \left\{\frac{p_t - \frac{s_t^L}{2}}{\Delta}\right\}\Delta \tag{A.6}$$

The second equality simply used the property of flooring function $\lfloor . \rfloor$. Note that $p_t - \frac{s_t^L}{2}$ is exactly the bid price under continuous pricing. Therefore, the widening effect on the bid side, i.e. the difference between B and $p_t - \frac{s_t^L}{2}$, is exactly $\left\{\frac{p_t - \frac{s_t^L}{2}}{\Delta}\right\}\Delta$. Let $z = \left\{\frac{p_t}{\Delta}\right\} = \frac{p_t}{\Delta} - \left\lfloor\frac{p_t}{\Delta}\right\rfloor$, we have the widening effect on the bid side as $\left\{\frac{p_t - \frac{s_t^L}{2}}{\Delta}\right\}\Delta = \left\{\frac{p_t}{\Delta} - \frac{s_t^L}{2\Delta}\right\}\Delta = \left\{z - \frac{s_t^L}{2\Delta}\right\}\Delta$, where the second equality used the fact that $\left\lfloor\frac{p_t}{\Delta}\right\rfloor$ is an integer and does not affect the fractional part of a number.

Since $z \sim U[0,1)$, we have $z - \frac{s_t^L}{2\Delta} \sim U\left[-\frac{s_t^L}{2\Delta}, 1 - \frac{s_t^L}{2\Delta}\right)$. Therefore, for any real

[13] In principle, any differentiable $f(p)$ with a bounded $f'(p)$ would lead to an approximately uniform distribution of $z_t = \{\frac{p_t}{\Delta}\}$, as long as the variation of p is much larger than a tick so that at any neighborhood of a specific p, $f(p)$ does not have a huge variation or a concentrated mass. This is arguably the case of the U.S. stock market where the stock prices span over \$1 to \$1000 while the tick size is at most one hundredth of the stock price.

number $\frac{s_t^L}{2\Delta}$, the fractional part of $z - \frac{s_t^L}{2\Delta}$ is uniformly distributed on $U[0,1)$. In other words, $\int_0^1 \left\{z - \frac{s_t^L}{2\Delta}\right\} \Delta \cdot dz = \frac{\Delta}{2}$ holds for any real number $\frac{s_t^L}{2\Delta}$. Therefore, the average widening effect on the bid side is $\frac{\Delta}{2}$.

The ask side is also subject to the same average widening effect of $\frac{\Delta}{2}$, which leads to a total widening effect Δ. ■

Proof of Proposition 3

From equation (6), we have:

$$\frac{2\sigma\lambda_J L\lambda_I v_t}{(\lambda_I v_t+\lambda_J p_t L)^2} = \frac{\Delta}{p_t^2} \tag{A.7}$$

Rearrange the terms, we have

$$\sqrt{\frac{2\sigma\lambda_J\lambda_I v_t}{\Delta L}} = \frac{\lambda_I v_t}{p_t L} + \lambda_J \tag{A.8}$$

$$p_t^* = (\sqrt{\frac{2\sigma\lambda_J L}{\lambda_I v_t \Delta}} - \frac{\lambda_J L}{\lambda_I v_t})^{-1}. \tag{A.9}$$

Next, we calculate the nominal spread s_t^{tot} under optimal pricing. Recall equation (3) that $s_t^{tot}(p_t) = \frac{2\sigma\lambda_J p_t^2 L}{\lambda_I v_t+\lambda_J p_t L} + \Delta = \frac{2\sigma\lambda_J p_t}{\frac{\lambda_I v_t}{p_t L}+\lambda_J} + \Delta$. Substituting the first term's denominator with equation (A.8) and the numerator with equation (A.9), we have:

$$s_t^{tot}(p_t^*) = \Delta/(1 - \sqrt{\frac{\Delta\lambda_J L}{2\sigma\lambda_I v_t}}) + \Delta. \tag{A.10}$$

Since L is small, the leading component of p_t^* is $(\sqrt{\frac{2\sigma\lambda_J L}{\lambda_I v_t \Delta}} - \frac{\lambda_J L}{\lambda_I v_t})^{-1} \rightarrow (\sqrt{\frac{2\sigma\lambda_J L}{\lambda_I \Delta v_t}})^{-1} = \sqrt{\frac{\lambda_I \Delta v_t}{2\sigma\lambda_J L}}$, and $s_t^{tot}(p_t^*) \rightarrow 2\Delta$. In other words, the nominal spread is two ticks when p_t is at its optimum, and the percentage spread $\mathcal{S}_t^{tot}$ is at its minimum. The residual parts originate from the informed traders' paid bid-ask spread to the market maker. Considering this small trunk of revenue source let the market maker to quote a slightly lower lot-driven spread in (A.10). The informed traders' volume is proportional to the nominal price, so the lot-driven spread becomes less sensitive to nominal price changes. Then, the equilibrium nominal price moves slightly higher to further reduce from tick-driven spread in (A.9).

Finally, we check whether p_t^* is the global minimum of $\mathcal{S}_t^{tot}$. Recall that $\mathcal{S}_t^{tot} = \frac{2\sigma\lambda_J p_t L}{\lambda_I v_t+\lambda_J p_t L} + \frac{\Delta}{p_t}$. We have $\frac{\partial^2 \mathcal{S}}{\partial p_t^2} = \frac{\partial}{\partial p_t}\left[\frac{2\sigma\lambda_J L\lambda_I v_t}{(\lambda_I v_t+\lambda_J p_t L)^2} - \frac{\Delta}{p_t^2}\right] = -\frac{4\sigma L\lambda_J^2\lambda_I}{(\lambda_I v_t+\lambda_J p_t L)^3} + \frac{2\Delta}{p_t^3}$. Therefore, $\frac{\partial^2 \mathcal{S}}{\partial P_t^2}$ may turn negative when p_t is large enough, hinting a possible local minimum of $\mathcal{S}_t^{tot}$ when $p_t \to \infty$. However, inserting $p_t \to \infty$ into equation (4) we get:

$$\mathcal{S}_t^{tot} = \frac{2\sigma\lambda_J p_t L}{\lambda_I v_t+\lambda_J p_t L} + \frac{\Delta}{p_t} \to 2\sigma. \tag{A.11}$$

This limit value of $\mathcal{S}_t^{tot}$ indicates that the market maker quotes the maximum possible spread $v_t \pm \sigma v_t$, which cannot be smaller than the minimum associated with $p_t = (\sqrt{\frac{2\sigma\lambda_J L}{\lambda_I v_t \Delta}} - \frac{\lambda_J L}{\lambda_I v_t})^{-1}$. Therefore, the $\mathcal{S}_t^{tot}$ reaches a global minimum if and only if $p_t = (\sqrt{\frac{2\sigma\lambda_J L}{\lambda_I v_t \Delta}} - \frac{\lambda_J L}{\lambda_I v_t})^{-1}$.[14] ■

Proof of Corollary 2

Since we have $s_t^{tot} = \frac{2\sigma\lambda_J p_t^2 L}{\lambda_I v_t} + \Delta$ with $L \to 0$, a further H-for-1 split changes the

[14] The problem is easier when $L \to 0$ and $\mathcal{S}_t^{tot} = \frac{2\sigma\lambda_J}{\lambda_I v_t} p_t L + \frac{\Delta}{p_t}$. Since $\frac{\partial^2 \mathcal{S}}{\partial p_t^2} = \frac{\partial}{\partial p_t}\left[-\frac{\Delta}{p_t^2}\right] = \frac{2\Delta}{p_t^3} > 0$, $p_t^* = \sqrt{\frac{\lambda_I \Delta v_t}{2\sigma\lambda_J L}}$ must be the unique global minimum.

lot-driven component from $s_t^L = \frac{2\sigma\lambda_J p_t^2 L}{\lambda_I v_t}$ to $\frac{2\sigma\lambda_J (p_t^2/H^2)L}{\lambda_I v_t}$ but does not change the tick-driven component Δ. Thus, observing a nominal spread s_t^{tot}, its lot-driven part $s_t^L = (s_t^{tot} - \Delta)$ will be changed to $(s_t^{tot} - \Delta)/H^2$, and the tick-driven component is still Δ. Therefore, our theory predicts the ex post nominal spread is $(s_t^{tot} - \Delta)/H^2 + \Delta$.[15] The nominal price also changes from p_t to p_t/H, so the percentage spread $\frac{s_t^{tot}}{p_t}$ will change to $\frac{(s_t^{tot}-\Delta)/H^2+\Delta}{p_t/H}$. The optimal H that minimizes $\frac{(s_t^{tot}-\Delta)/H^2+\Delta}{p_t/H}$ is $\sqrt{\frac{s_t^{tot}-\Delta}{\Delta}}$, which is only dependent on the ratio of the observed spread s_t^{tot} and the tick size Δ. ■

Proof of Corollary 3

Let $L \to 0$ in equations (A.9) and (A.10), and we immediately have $p_t^* \to \sqrt{\frac{\lambda_I \Delta v_t}{2\sigma\lambda_J L}}$ and $s_t^{tot}(p_t^*) \to 2\Delta$. Therefore, $\mathcal{S}(p_t^*) = \frac{2\Delta}{p_t^*} = \sqrt{\frac{8\sigma\lambda_J \Delta L}{\lambda_I v_t}}$, which is proportional to $\sqrt{\Delta L}$ .Thus, although the optimal nominal spread $s_t^{tot}(p_t^*)$ is 2Δ and is not dependent on firm fundamentals, the optimal percentage spread does. Intuitively, more volatile firms ($\sigma\lambda_J$) need to choose lower prices to incentivize the market makers to quote the two-tick spread. On the other hand, firms with higher latent liquidity demand (λ_I) and larger market cap (v_t) can choose higher nominal prices to reach the 2-tick nominal spread. ■

Proof of Corollary 4

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[15] Again, we do not need to observe the firm fundamentals (σ, v_t, λ_I, and λ_J) to calculate the spread changes due to a stock split, because the observed spread, s_t^{tot}, is their sufficient statistics in determining the split ratio.

In Proposition 3, we have $s_t^{tot}(p_t) = \frac{2\sigma\lambda_J p_t^2}{\lambda_I v_t} L + \Delta$. The percentage bid-ask spread $\mathcal{S}_t^{tot} = \frac{s_t^{tot}(p_t)}{p_t} = \frac{2\sigma\lambda_J}{\lambda_I v_t} L p_t + \frac{\Delta}{p_t}$ depend on the firm's choice of p_t. Inserting the proportional lot size $\mathbb{L}(p_t) = k^L / p_t$, we have:

$$\mathcal{S}_t^{tot}(p_t) = \frac{2\sigma\lambda_J}{\lambda_I v_t} k^L + \frac{\Delta}{p_t}. \qquad \text{(A.12)}$$

(A.12) indicates that the seemingly flexible proportional lot size imposed a unified dollar lot size k^L to all stocks, and the lot-driven component is dependent only on k^L but not on p_t. In other words, the firms can't adjust their nominal prices to lower the market makers' adverse selection costs, and their choice of nominal prices affect only the relative tick size. Therefore, the percentage spread $\mathcal{S}_t^{tot} = \frac{s_t^{tot}(p_t)}{p_t} = \frac{2\sigma\lambda_J}{\lambda_I v_t} k^L + \frac{\Delta}{p_t}$ monotonically decreases with p_t. The proportional lot size essentially removes one side of the tick/lot trade-off and encourages $p_t^* \to \infty$, where $\mathcal{S}_t^{tot,*} = \frac{2\sigma\lambda_J}{\lambda_I v_t} k^L$.

On the other hand, if we insert the proportional lot size $\mathbb{D}(p_t) = k^\Delta p_t$ into $s_t^{tot}(p_t) = \frac{2\sigma\lambda_J p_t^2}{\lambda_I v_t} L + \Delta$, we have:

$$\mathcal{S}_t^{tot} = \frac{s_t^{tot}(p_t)}{p_t} = \frac{2\sigma\lambda_J p_t}{\lambda_I v_t} L + k^\Delta. \qquad \text{(A.13)}$$

(A.13) indicates that the proportional tick size system imposed a unified relative tick size k^Δ to all stocks. No firms can reduce their percentage bid-ask spread lower than k^Δ. With uniform lot size and proportional tick size, the percentage spread $\mathcal{S}_t^{tot} = \frac{s_t^{tot}(p_t)}{p_t} = \frac{2\sigma\lambda_J p_t}{\lambda_I v_t} L + k^\Delta$ monotonically increases with p_t. The proportional tick size essentially removes the other side of the tick/lot trade-off and encourages $p_t^* \to 0$, where $\mathcal{S}_t^{tot,*} = k^\Delta$.

Similarly, when both proportional tick and lot size systems are implemented, we have:

$$s_t^{tot}(p_t) = \frac{2\sigma\lambda_J p_t}{\lambda_I v_t} k^L + k^\Delta p_t \tag{A.14}$$

$$\mathcal{S}_t^{tot} \equiv \frac{s_t^{tot}(P_t)}{p_t} = \frac{2\sigma\lambda_J}{\lambda_I v_t} k^L + k^\Delta. \tag{A.15}$$

Under the fixed Δ and L system, firms adjust their nominal price to choose their optimal dollar lot size and relative tick size. In equilibrium, the balance is reached with $s_t^{tot}(p_t^*) = \frac{2\sigma\lambda_J (p_t^*)^2}{\lambda_I v_t} L + \Delta = \Delta + \Delta = 2\Delta$. (A.15) shows that the proportional tick and lot system is one-size-fit-all: it imposes a unified dollar lot size and relative tick size to all stocks. Next, we show that such system harms liquidity provision if k^L and k^Δ is selected using any representative stock.

We denote $\chi(p_t) = \frac{p_t}{p_\Omega}$ as the distance between the representative price p_Ω and a stock priced at p_t. For a stock priced at p_t, its new tick size is χ times Δ, while its new lot size becomes χ^{-1} times L. Since the tick- (lot-) driven percentage spread is proportional to the tick (lot) size, the new nominal spread is $s_t^{tot}(p_t) = (\chi^{-1} + \chi)\Delta$. Observe that $(\chi^{-1} + \chi)\Delta \geq 2\Delta$, where the equality holds only if $p_t = p_\Omega$ (i.e., its tick and lot sizes are unchanged). The bid-ask spread widens for all stocks with $p_t \neq p_\Omega$.■