An Economic Model of a Decentralized Exchange with Concentrated Liquidity*

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Abstract

We develop a model of a decentralized exchange that allows investors to concentrate liquidity within pre-specified price intervals (e.g., Uniswap V3). Providing liquidity for a risky/risk-free asset pair within any interval is analogous to investing in a dynamic portfolio of those assets, subject to an arbitrage cost, where the risky asset weight declines as its price increases. We derive equilibrium liquidity provision for each price interval and provide a simple approximation that can be useful for empirical work. We also show that liquidity provision generates an ex-fee return approximately equivalent to the return from a covered call trading strategy.

Keywords: Decentralized Exchange, DEX, Automated Market Makers, AMM, Concentrated Liquidity, Uniswap V3

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A decentralized exchange (DEX) is an innovation that allows investors to exchange digital assets through the use of smart contracts deployed on a blockchain. The first successful design of a DEX (e.g., Uniswap V1/V2) allows investors to passively provide liquidity uniformly across all price levels. While innovative, this design suffers from inefficiencies because uniform liquidity provision results in a substantial amount of liquidity being provided at price levels that are unlikely to be reached. To overcome these inefficiencies, a new design for liquidity provision, *concentrated liquidity provision*, has been developed with this new design being characterized by allowing investors to provide liquidity for only trades with trade prices within particular price intervals explicitly specified by the liquidity provider. The aim of this paper is to shed light on the referenced innovation by studying the equilibrium liquidity provision for a DEX with concentrated liquidity.

Formally, our paper puts forth an economic model of a DEX with concentrated liquidity provision (e.g., Uniswap V3). In providing such a model, we characterize the investment return profile for concentrated liquidity providers, and also provide a useful expression for equilibrium liquidity provision distribution across all price levels. Our analysis of the investment return to liquidity providers offers worthwhile context for investors deciding whether to provide liquidity to a DEX such as Uniswap V3. Moreover, our characterization of equilibrium liquidity provision provides an empirically useful framework that can be utilized to study DEX liquidity provision in practice.

To provide more detail, we examine a continuous time model with a single DEX that facilitates trading of a risky asset, hereafter ETH, against a risk-free asset, hereafter USDC.[1] Our model consists of investors with identical investment horizons and traders with exogenous trading demand. All investors have access to the risk-free asset, USDC, and can lend and borrow that asset at an exogenous risk-free rate. At the beginning of the investment horizon, investors optimally allocate capital across DEX liquidity provision and a portfolio of

[1]ETH represents *ether*, the native cryptoasset of the Ethereum blockchain (a risky crypto-asset) while USDC represents USD coin, a stable coin pegged to the US dollar. ETH-USDC is typically the most actively traded token pair at decentralized exchanges such as Uniswap.

risky assets. Thereafter, traders arrive sequentially and trade at the DEX. Each trader pays a proportional fee on her trading volume. These fees are distributed pro-rata to the investors who provided liquidity to the price interval that contains the trades. At the conclusion of the investment horizon, all investors liquidate their investments and realize their pay-offs. We assume that, over the investment horizon, ETH-USDC prices at non-DEX venues (e.g., at centralized exchanges) follow an exogenous generalized diffusion process which reflects innovations in public information. In contrast, ETH-USDC prices at the DEX follow a mechanical pricing function known as a Constant Product Automated Market Maker (CPAMM) function (see John et al. 2023 for details). We assume that arbitrage traders immediately exploit any price dislocations between the DEX and non-DEX trading venues, thereby maintaining alignment between ETH-USDC prices at the DEX and the ETH-USDC price at non-DEX venues in equilibrium.

Crucially, our model departs from prior literature by allowing for concentrated liquidity provision at the DEX. More explicitly, as per Uniswap V3, we assume that the DEX partitions the range of ETH-USDC prices into intervals and that investors may select any subset of price intervals to which they can provide liquidity. Our model contrasts with prior work that assumes that any DEX liquidity provision must necessarily apply uniformly across all price levels. Note that early DEX deployments (e.g., Uniswap V1/V2) impose this latter condition but recent DEX deployments (e.g., Uniswap V3) allow for concentrated liquidity provision as per our model.

Our paper consists of three sets of results. First, we provide results that characterize the investment return profile for providing liquidity to an arbitrary price interval. In particular, we demonstrate that ex-fee returns from liquidity provision are equivalent to the returns from investing in a dynamic ETH-USDC portfolio minus the loss-versus-rebalancing effect first highlighted by Milionis et al. (2022). Notably, we explicitly derive the dynamic portfolio weighting and demonstrate that the ETH portfolio weight is decreasing in the ETH-USDC price (Proposition 4.2). After characterizing the returns from liquidity provision for any given interval, our second set of results compares those returns across arbitrary intervals. Our key insight here is that the ex-fee return from providing liquidity to a price interval lying above the current ETH-USDC price increases as we consider intervals further above the ETH-USDC price. Further, we show that ex-fee returns from liquidity provision to all such intervals lying above the current ETH-USDC price are always dominated by the return from investing only in ETH (Propositions 4.3 and 4.4). Similarly, the ex-fee return from providing liquidity to a price interval lying below the current ETH-USDC price increases as we consider intervals further below the ETH-USDC price and ex-fee returns for all such intervals are always dominated by the return from holding USDC (Propositions 4.3 and 4.5). This second set of results implicitly highlights that equilibrium fee returns must be sufficiently large to offset the opportunity cost of not investing directly in ETH or USDC. Relatedly, our third set of results, which constitute our primary contribution, provides a closed-form expression for the equilibrium liquidity provision distribution across price levels. In more detail, after characterizing the equilibrium liquidity provision for any arbitrary width of the price interval, we then study a limiting case whereby the price intervals collapse to zero width, which allows us to derive closed-form expressions for both equilibrium liquidity provision (Proposition 4.7) and equilibrium liquidity provision returns (Proposition 4.8). Notably, our equilibrium liquidity provision expression is easy to compute, and we show how it can be used to approximate liquidity provision in practice. Our results can thus be applied to empirically test the distribution of liquidity provision at DEXs that support concentrated liquidity. Importantly, we also demonstrate that our derived equilibrium liquidity provision return is similar to the return on an ETH-USDC covered call trading strategy, clarifying that concentrated liquidity provision differs from investing directly in ETH mainly by foregoing ETH price appreciation beyond a certain point.

In order to explain our findings, we emphasize that providing DEX liquidity to a given price interval comes with the restriction that the liquidity can be utilized to facilitate trades only within that interval. Importantly, in exchange for providing such inventory, each investor receives 1.) a pro-rata share of any trading fees earned from trades executed within that interval and 2.) a pro-rata share of the total inventory associated with the selected price interval. We show that the return from the pro-rata share of inventory is equivalent to the return on the portfolio of ETH-USDC inventory for the interval minus the loss-versus-rebalancing cost of Milionis et al. (2022). We characterize this aforementioned ETH-USDC inventory portfolio for each interval (Proposition 4.1), hereafter referred to as the *liquidity portfolio* for the given price interval, noting that this characterization is a specialization of a known result that applies to a more general class of AMMs (see Milionis et al. 2022). Importantly, we demonstrate that the liquidity portfolio possesses dynamic portfolio weights which vary with the ETH-USDC price (Proposition 4.2). In particular, we show that the ETH portfolio weight is unity when the ETH-USDC price is below the price interval, zero when the ETH-USDC price is above the price interval, and continuously decreasing from unity to zero as the ETH-USDC price increases through the price interval. It is important to note that Proposition 4.2 would not arise under uniform liquidity provision. More explicitly, a DEX with a CPAMM function and uniform liquidity provision (e.g., Uniswap V1 and V2) generates a 50-50 portfolio for the two assets being traded against each other (see Angeris et al. 2021). Crucially, the referenced 50-50 portfolio weighting for uniform liquidity provision is static and invariant to market prices; this static nature of the liquidity portfolio under uniform liquidity provision contrasts with the dynamic nature of the liquidity portfolio under concentrated liquidity provision.

In order to understand the dynamic nature of the liquidity portfolio (i.e., Proposition 4.2), it is necessary to recognize that the mechanical nature of DEX pricing implies that the ETH-USDC price at the DEX does not directly respond to public information. Rather, when public information induces an increase in the exogenous ETH-USDC price at non-DEX venues (e.g., at centralized exchanges), this generates profitable trading opportunities at the DEX whereby arbitrage traders buy ETH at a stale price from the DEX and sell ETH at the new higher price at a non-DEX venue (see Capponi and Jia 2021). The referenced arbitrage trading ensures that the DEX price is always aligned with the ETH-USDC price at non-DEX venues. Thus, as the ETH-USDC price increases through a particular price interval, arbitrage activities lead to net buying of ETH at the DEX; in turn, since an ETH buy at a DEX is implemented as a swap of ETH for USDC, such net buying decreases the ETH portfolio weight of the liquidity portfolio for that price interval.

Our second set of results provides comparative insights for the investment returns from liquidity provision at a DEX. Most notably, Proposition 4.3 establishes that the ex-fee realized return for providing liquidity at a DEX to *any* price interval is always dominated by the return from investing in either ETH or USDC directly. More precisely, the ex-fee realized return from providing liquidity to an interval above the initial ETH-USDC price is dominated by investing directly in ETH, whereas the ex-fee realized return from providing liquidity to an interval below the initial ETH-USDC price is dominated by directly holding USDC. Of note, Proposition 4.3 can be understood as a formal demonstration of the loss-versus-holding (a.k.a. impermanent loss) concept to a CPAMM DEX with concentrated liquidity.

Importantly, Proposition 4.3 helps us frame the trade-off that liquidity providers internalize when providing liquidity to a given price interval. More explicitly, since the return on a liquidity portfolio is always dominated by the return from holding ETH (USDC) directly, it follows that an investor will provide liquidity to the DEX within a given price interval only if the expected return from trading fees is high enough to off-set the opportunity cost of accepting a lower return from the liquidity portfolio. Crucially, the fact that investors earn their pro-rata share of fee revenues from trading that occurs within their price interval implies that the liquidity provision to a given interval will adjust in equilibrium to guarantee that the return from fees exactly offsets the opportunity cost of liquidity provision.

Given all the aforementioned context, we turn to our main result which is to provide the equilibrium liquidity provision for each price interval. We specifically provide that result in two ways. First, when deriving our equilibrium model solution (Proposition 3.1), we provide a general expression for equilibrium liquidity provision for each price interval. Subsequently, in Proposition 4.7, we derive equilibrium liquidity provision in a limiting case of our model whereby the width of the price interval goes to zero and apply that limit to construct a simple approximation of equilibrium liquidity provision for each price interval. We offer both the general expression and an approximate expression because the general result of Proposition 3.1 is somewhat opaque, whereas the limiting result of Proposition 4.7 has an intuitive form. More precisely, the limiting equilibrium liquidity provision for each price interval has quotient form with the numerator representing the expected fee revenue and the denominator representing the extent to which ex-fee expected returns fall short of risk-adjusted returns from investing directly in ETH. This limiting expression thus transparently highlights the manner in which equilibrium liquidity provision increases with fees but decreases with the opportunity cost of investment. Notably, we demonstrate that our limiting distribution of liquidity provision can be utilized to form a simple approximation of equilibrium liquidity provision that is easy to compute across all price intervals and commensurately easy to apply for future empirical work and comparative statics.

We conclude our analysis by offering an intuitive characterization of the returns to liquidity provision at a DEX with concentrated liquidity. More explicitly, we conclude with Proposition 4.8, a result which reveals that providing liquidity to a particular price interval at a DEX with concentrated liquidity is approximately equivalent to holding ETH and shorting a particular ETH-USDC call against that ETH position (i.e., a covered call). The aforementioned call option is characterized by a strike price which is within the price interval to which liquidity is being provided and a maturity equal to the end of the investor's investment horizon. To understand this result, recall from Proposition 4.2 that the liquidity portfolio for a given price interval consists of only ETH when the ETH-USDC price is below the price interval and the liquidity portfolio consists of only USDC when the ETH-USDC price is above the interval. In turn, as a consequence, providing liquidity to a price interval over a fixed investment horizon corresponds to receiving the ETH pay-off if the ETH-USDC price ends below the price interval but otherwise corresponds to the pay-off from having sold the ETH inventory for USDC if the ETH-USDC price ends above the price interval. Importantly, as per Proposition 4.8, this pay-off profile can be approximated by the aforementioned covered call strategy. In particular, under the aforementioned covered call strategy, if the terminal ETH-USDC price ends below the interval, then the call option expires worthless and the investor holds only ETH, just as if the investor had provided liquidity to the DEX. In contrast, when the terminal ETH-USDC price ends above the interval, then the call option is optimally exercised and therefore the investor sells her ETH in return for USDC which also generates the same pay-off in this case from providing liquidity to the DEX.

In broad terms, our paper relates to the literature examining the economics of blockchain. Makarov and Schoar (2022), John et al. (2022), and John et al. (2023) provide surveys of that literature. That literature includes many strands of work including blockchain economic security (e.g., Biais et al. 2019, Saleh 2021 and Chiu and Koeppl 2022), blockchain microstructure elements (e.g., Easley et al. 2019, Huberman et al. 2021 and Lehar and Parlour 2020), smart contracts (e.g., Cong and He 2019) and tokenomics (e.g., Cong et al. 2021 and Mayer 2022). More recently, a literature examining Decentralized Finance (DeFi) applications on blockchain has emerged, and our work contributes especially to that strand of work. In more detail, the DeFi literature particularly examines lending platforms (see, e.g., Chiu et al. 2022, Lehar and Parlour 2022, Chaudhary et al. 2023, Rivera et al. 2023) and DEXs, whereby our contribution is to the latter.

The literature on DEXs is young but quickly growing. The early literature on decentralized exchanges includes Aoyagi (2020), Aoyagi and Ito (2021), Capponi and Jia (2021), Lehar and Parlour (2021), Park (2021), Hasbrouck et al. (2022) and Milionis et al. (2022). Apart from Milionis et al. (2022), all the aforementioned papers focus on settings with uniform liquidity provision as per the practice of many DEX deployments (e.g., Uniswap V1 and V2). Thus, our contribution relative to those works is that we study a new type of DEX, namely a DEX with concentrated liquidity provision. It is noteworthy that Milionis et al. (2022) study a more general setting but focus on establishing a generic cost of DEX investing known as Loss-Versus-Rebalancing (LVR) rather than examining specific features of any particular type of DEX. Importantly, the main insight of Milionis et al. (2022) holds even out-of-equilibrium and thus Milionis et al. (2022) study a model with an exogenous level of liquidity provision. In contrast, we determine liquidity provision endogenously and a primary contribution of our work is to provide a simple expression for equilibrium liquidity provision across all price intervals.

While we are the first to study an equilibrium model of a DEX with concentrated liquidity provision, there exist other papers that either examine this specific setting empirically or out-of-equilibrium (i.e., with exogenous liquidity provision). With regard to empirical work, complementary to our work is the work of Barbon and Ranaldo (2022), Lehar et al. (2022) and Caparros et al. (2023), both of which conduct empirical analysis on Uniswap V3 (a DEX that allows for concentrated liquidity provision). Of note, Lehar et al. (2022) also provide theoretical analysis but focus on competition across markets, abstracting from liquidity provision across multiple intervals, rather than examining the implications of concentrated liquidity provision within a single market. With regard to works also studying a DEX with concentrated liquidity provision in a theoretical context, Neuder et al. 2021 and Heimbach et al. 2022 theoretically examine the return profile for a single investor. Our work differs from those works in that we provide an equilibrium analysis with endogenously derived liquidity provision whereas the referenced papers abstract from equilibrium asset pricing conditions and also take DEX liquidity provision as exogenous.

1 Institutional Detail Regarding Uniswap V3

Before stating our formal economic model, we first clarify the mechanics of a DEX with concentrated liquidity. More explicitly, within this section, we explain the mechanics of the most prominent DEX that offers concentrated liquidity provision, Uniswap V3. As an aside, our model exposition in Section 2 is largely self-contained so that a reader may skip this section with minimal loss in clarity with regard to our formal analysis.

The Uniswap V3 specification governs exchanges between two cryptoassets. For expositional simplicity, we assume the two cryptoassets are ETH (a risky cryptoasset) and USDC (a USD stablecoin), with prices stated in terms of USDC per ETH token. The price space is partitioned into a set of intervals. Each price interval corresponds to the range defined by adjacent values on a price grid which is given as follows:

$$\Psi_k = (1 + \Delta)^k \tag{1}$$

for all $k \in \mathbb{Z}$ and with $\Delta > 0$ determining the geometric width of each price interval (i.e., $\frac{\Psi_{k+1}}{\Psi_k} = 1 + \Delta$ for all i).

Associated with each interval is a portfolio of ETH and USDC contributed by liquidity suppliers. The ETH and USDC in this portfolio are available for exchange in that ETH buyers provide USDC as payment in return for ETH, whereas ETH sellers provide ETH in return for USDC as payment. To provide more detail regarding the mechanics of trades, Uniswap V3 determines pricing by requiring the following invariant hold at all times:

$$\left(ETH_{i,t} + \frac{L_i}{\sqrt{\Psi_{i+1}}}\right)\left(USDC_{i,t} + L_i\sqrt{\Psi_i}\right) = L_i^2 \tag{2}$$

where $ETH_{i,t}$ denotes the ETH inventory in price interval i at time t, $USDC_{i,t}$ denotes the USDC inventory in price interval i at time t and $L_i \geqslant 0$ denotes an endogenous market-determined quantity, generally termed "liquidity" by practitioners.

To see how Equation (2) determines pricing, note that trading δ_{ETH} ETH alters the ETH inventory from $ETH_{i,t}$ to $ETH_{i,t} - \delta_{ETH}$ where we use the convention that $\delta_{ETH} > 0$ corresponds to an ETH buy while $\delta_{ETH} < 0$ corresponds to an ETH sell. Importantly, by altering the ETH inventory level, trading ETH alters the first term on the left hand side of Equation (2) and thus requires an offsetting adjustment to the second term on the left hand side of Equation (2) (i.e., to USDC inventory) so as to maintain Equation (2) after the trade.

In more detail, an ETH buy (i.e., $\delta_{ETH} > 0$) reduces ETH inventory and thereby requires an off-setting increase in USDC inventory of $\delta_{USDC} > 0$, whereas an ETH sale (i.e., $\delta_{ETH} < 0$) reduces ETH inventory and thereby requires an off-setting decrease of USDC inventory by $\delta_{USDC} < 0$. More formally, an ETH trade not only alters ETH inventory to $ETH_{i,t} - \delta_{ETH}$ but, to maintain Equation (2), it must be accompanied by an alteration in USDC inventory to $USDC_{i,t} + \delta_{USDC}$ where δ_{USDC} can be derived by imposing the invariant in Equation (2) with ETH and USDC inventory levels updated to those after the trade:

$$\left(ETH_{i,t} - \delta_{ETH} + \frac{L_i}{\sqrt{\Psi_{i+1}}}\right)\left(USDC_{i,t} + \delta_{USDC} + L_i\sqrt{\Psi_i}\right) = L_i^2 \tag{3}$$

When $\delta_{ETH} > 0$, the additional USDC inventory of $\delta_{USDC} > 0$ is deemed as the payment for the ETH buy. Similarly, when $\delta_{ETH} < 0$, the reduction in USDC inventory is deemed as the proceeds from the ETH sale. In turn, given that interpretation, it is easy to compute the (average) price of an ETH trade of δ_{ETH} by solving for δ_{USDC} and then taking the price as the amount of USDC per unit ETH (i.e., $\frac{\delta_{USDC}}{\delta_{ETH}}$). More explicitly, Equation (2) and (3) collectively imply the following average price, $P_t^{DEX}(\delta_{ETH})$, for trading δ_{ETH} ETH:[2]

$$P_t^{DEX}(\delta_{ETH}) = \frac{USDC_{i,t} + L_i\sqrt{\Psi_i}}{ETH_{i,t} + \frac{L_i}{\sqrt{\Psi_{i+1}}} - \delta_{ETH}}, \qquad \delta_{ETH} \in [\delta_{i,t}^-, \delta_{i,t}^+] \tag{4}$$

where δ_{ETH} is restricted to the domain $[\delta_{i,t}^-, \delta_{i,t}^+]$ with $\delta_{i,t}^- < 0$ representing the largest ETH sale feasible at time t within price interval i and $\delta_{i,t}^+ > 0$ representing the largest ETH buy feasible at time t within price interval i. To provide further context, the largest feasible ETH buy is the ETH quantity that would fully deplete the ETH inventory within the price interval (i.e., $\delta_{i,t}^+ = ETH_{i,t}$), whereas the largest feasible ETH sale is the ETH sale that would fully deplete the USDC inventory within the interval (i.e., $|\delta_{i,t}^- \times P_t^{DEX}(\delta_{i,t}^-)| \leqslant USDC_{i,t}$). If a

[2]Note that Equation (4) is identical to the pricing for uniform liquidity provision derived in John et al. (2023) except that the true inventory levels, $ETH_{i,t}$ and $USDC_{i,t}$, are inflated by additive factors. More explicitly, $ETH_{i,t}$ is replaced by $ETH'_{i,t} = ETH_{i,t} + \frac{L_i}{\sqrt{\Psi_{i+1}}}$ whereas $USDC_{i,t}$ is replaced by $USDC'_{i,t} = USDC_{i,t} + L_i\sqrt{\Psi_i}$. In practice, $ETH'_{i,t}$ and $USDC'_{i,t}$ are generally referred to as "virtual" inventory.

trader wishes to place an order larger than $\delta_{ETH} \in [\delta_{i,t}^-, \delta_{i,t}^+]$, then the maximum feasible trade is executed within the price interval i, and the remainder of the trading volume is executed within other price intervals. More explicitly, if the trader wishes to trade $\delta > \delta_{i,t}^+$, then a trade size of $\delta_{i,t}^+$ is executed within price interval i and the remainder of the trade is executed within price intervals above price interval i; similarly, if the trader wishes to trade $\delta < \delta_{i,t}^-$, then a trade size of $\delta_{i,t}^-$ is executed within price interval i and the remainder of the trade is executed within price intervals below price interval i.

Crucially, note that, as per Equation (4), trading alters the ETH-USDC price in the direction of the trade with an ETH buy increasing the ETH-USDC price and an ETH sell decreasing the ETH-USDC price (i.e., $\frac{dP}{d\delta} > 0$ in Equation 4). Uniswap V3 is particularly specified such that the ETH-USDC price moves continuously upwards through a price interval due to ETH buying; the buying depletes the ETH inventory, and the ETH-USDC price enters the adjacent upper interval exactly when the initial interval possesses zero ETH inventory. Similarly, the ETH-USDC price moves continuously downward through a price interval due to ETH selling; the selling depletes the USDC inventory, and the ETH-USDC price enters the adjacent lower interval exactly when the initial interval possesses zero USDC inventory.

Given L_i, Uniswap V3 is simply a mechanical rule. There is no presumption that the rule represents an optimal market structure. Nonetheless, L_i is not an exogenous parameter; rather, it is an endogenous economic quantity determined by the level of investment from liquidity providers. An important contribution of our work is that we depart from prior literature by deriving L_i as an equilibrium object rather than taking it as exogenous.

2 A Model of Concentrated Liquidity Provision

We model a single investment horizon from time $t = 0$ to $t = T$. At $t = 0$, investors arrive and allocate their capital across all available investment opportunities. At $t = T$, investors liquidate their investments and realize their pay-offs. We assume that investors select their portfolios at $t = 0$ to maximize their expected utility.

2.1 Assets

There exist two assets: a risk-free asset (USDC) and a risky asset (ETH). USDC is the numeraire and may be borrowed or lent at the exogenous risk-free rate $r > 0$. In contrast, ETH is a risky asset with ETH-USDC prices $\{P_t\}_{t=0}^T$ evolving according to an exogenous continuous time diffusion process given by:

$$\frac{dP_t}{P_t} = r\ dt + \sigma_t\ dB_t^{\mathbb{Q}} \tag{5}$$

where $\{B_t^{\mathbb{Q}}\}_{t=0}^T$ denotes a Brownian motion under the risk-neutral measure $\mathbb{Q}$ while $\{\sigma_t\}_{t=0}^T$ denotes a non-negative process for instantaneous ETH return volatility. We require that $\{\sigma_t\}_{t=0}^T$ is such that $\{P_t\}_{t=0}^T$ is non-negative, fully supported on $\mathbb{R}_+$ and further that there exists a continuous function $f(p,t)$ which gives the density of $p_t := \log(P_t)$ at value p; we also require $\mathbb{E}[e^{\frac{1}{2}\int_0^T \sigma_t^2 dt}] < \infty$. Note that all these regularity conditions are satisfied by geometric Brownian motion (i.e., $\sigma_t = \sigma > 0$), the most common special case of Equation (5).

2.2 Decentralized Exchange (DEX)

We model a single Decentralized Exchange (DEX) which allows for the trading of ETH against USDC and operates as described in Section 1. Investors may invest in the DEX by providing liquidity to the DEX which facilitates the DEX's trading activity. In more detail, an investor providing liquidity to the DEX means that the investor provides the DEX with ETH and USDC inventory which is then used by the DEX to meet demand for traders buying or selling ETH against USDC.

The DEX partitions the feasible range of ETH-USDC prices into exogenous intervals and each investor may concentrate her liquidity provision on any subset of those intervals. Providing liquidity to a particular price interval implies that the investor's inventory can be used for trading at the DEX only if that trading occurs at ETH-USDC prices within that specific price interval. In turn, an investor providing liquidity to a particular price interval does not improve liquidity for traders when ETH-USDC prices are outside that price interval.

We let price interval $i \in \mathbb{Z}$ correspond to interval $[\Psi_i, \Psi_{i+1}]$ where, as in practice, each interval endpoint is given explicitly by $\Psi_k = (1+\Delta)^k$ with $\Delta > 0$ determining the geometric width of each price interval (i.e., $\frac{\Psi_{k+1}}{\Psi_k} = 1 + \Delta$ for all k). Then, the gross return from providing liquidity to interval i, $R_{DEX,i}$, is given as follows:

$$R_{DEX,i} = R^i_{P\&L} + \phi_i \tag{6}$$

where $R^i_{P\&L}$ denotes the ex-fee gross return on the liquidity providers inventory for price interval i and ϕ_i denotes the fees accrued by liquidity providers within price interval i for providing a unit of inventory capital to price interval i. We subsequently clarify how $R^i_{P\&L}$ and ϕ_i are each determined.

2.2.1 Ex-Fee Return to Liquidity Providers, $R^i_{P\&L}$

As noted earlier, the liquidity provided to the DEX for any particular price interval is provided as inventory in the form of ETH and USDC and thus the liquidity provided for any particular price interval constitutes a portfolio of ETH and USDC. Notably, when an investor provides liquidity at a particular price interval, she becomes a pro-rata owner of the portfolio associated with that price interval, which we refer to as the liquidity portfolio for that price interval. In turn, the ex-fee gross return to an investor for providing liquidity to price interval i, $R^i_{P\&L}$, is the gross liquidity portfolio return, which is given explicitly as follows:

$$R^i_{P\&L} = \frac{\Pi_{i,T}}{\Pi_{i,0}} \tag{7}$$

with $\Pi_{i,t}$ denoting the liquidity portfolio value of inventory associated with price interval i. Since the liquidity portfolio consists of only ETH and USDC, $\Pi_{i,t}$ is the sum of the market value of ETH and USDC in the portfolio, which is given explicitly as follows:

$$\Pi_{i,t} = USDC_{i,t} + ETH_{i,t} \times P_t \tag{8}$$

where $USDC_{i,t}$ denotes the inventory of USDC within price interval i at time t, and $ETH_{i,t}$ denotes the inventory of ETH within price interval i at time t.

The value of inventory, $\Pi_{i,t}$, fluctuates not only due to fluctuations of ETH-USDC prices (i.e., changes in P_t) but also due to changes in the quantity of ETH and USDC associated with the price interval (i.e., changes in $ETH_{i,t}$ and $USDC_{i,t}$). In particular, trading at the DEX, when prices are within price interval i, leads to changes in the quantity of ETH and USDC associated with price interval i. For example, buying ETH against USDC at a DEX entails removing ETH inventory from the DEX in exchange for depositing USDC inventory to the DEX, with the quantity of USDC deposited corresponding to the dollar price paid for the ETH removed. As per Uniswap V3, we assume that the DEX employs a a Constant Product Automated Market Maker (CPAMM) for pricing (see Section 1). We also assume that the ETH-USDC prices at the DEX remain aligned with the true value of ETH-USDC prices due to arbitrage trading and we follow Milionis et al. (2022) by abstracting from arbitrageurs trading fees. In that case, the quantity of USDC, $USDC_{i,t}$, and the quantity of ETH, $ETH_{i,t}$, in price interval i at time t are given explicitly as follows:[3]

$$USDC_{i,t} = \left(\sqrt{\tilde{P}_{i,t}} - \sqrt{\Psi_i}\right) \times L_i, \qquad ETH_{i,t} = \left(\frac{1}{\sqrt{\tilde{P}_{i,t}}} - \frac{1}{\sqrt{\Psi_{i+1}}}\right) \times L_i \tag{9}$$

where L_i is an endogenous quantity that practitioners refer to as the "liquidity" for price interval i that is proportional to the dollar value of the liquidity provided to interval i, while $\tilde{P}_{i,t}$ denotes the projection of the ETH-USDC price onto interval i given by:

[3]Formally, Equation (9) follows directly from the equations that define the Uniswap V3 protocol, Equations (2) and (4), when imposing the additional requirement that the marginal ETH-USDC DEX price aligns with the price at other trading venues (i.e., $\lim_{\delta \to 0^+} P_t^{DEX}(\delta) = P_t$ for all t where $P_t^{DEX}(\delta)$ is defined explicitly in Equation 4).

$$\tilde{P}_{i,t} = \begin{cases} \Psi_{i+1} & \text{if } P_t > \Psi_{i+1} \\ P_t & \text{if } P_t \in [\Psi_i, \Psi_{i+1}] \\ \Psi_i & \text{if } P_t < \Psi_i \end{cases} \tag{10}$$

Equations (7) - (10) then imply that $R^i_{P\&L}$ is given explicitly as follows:

$$R^i_{P\&L} = \frac{\left(\sqrt{\tilde{P}_{i,T}} - \sqrt{\Psi_i}\right) + \left(\frac{1}{\sqrt{\tilde{P}_{i,T}}} - \frac{1}{\sqrt{\Psi_{i+1}}}\right) \times P_T}{\left(\sqrt{\tilde{P}_{i,0}} - \sqrt{\Psi_i}\right) + \left(\frac{1}{\sqrt{\tilde{P}_{i,0}}} - \frac{1}{\sqrt{\Psi_{i+1}}}\right) \times P_0} \tag{11}$$

2.2.2 Fees Accrued to Liquidity Providers, ϕ_i

Turning to fees accrued by liquidity providers, the DEX charges a proportional fee, $\phi \geqslant 0$, on all trading volume. All fees generated from trading within price interval i are paid pro-rata to the investors who provide liquidity for that interval. More formally, letting $V > 0$ denote the trading volume per unit time and letting $\mathcal{I}(\cdot)$ denote an indicator variable, the cumulative fees accrued for price interval i, Φ_i, from time 0 to time T is given as follows:[4]

$$\Phi_i = \int_0^T \phi \times V \times \mathcal{I}(P_t \in [\Psi_i, \Psi_{i+1}]) \, dt \tag{12}$$

Furthermore, those total fees, Φ_i, are distributed pro-rata among the liquidity providers for the price interval i. Since the total investment by liquidity providers is given by the portfolio value of assets associated with the price interval at $t = 0$, the fees accrued for a unit of investment capital to price interval i, ϕ_i, is therefore given explicitly as follows:

$$\phi_i = \frac{\Phi_i}{\Pi_{i,0}} \tag{13}$$

[4]Note that we can easily incorporate the fact that higher inventory in any given interval leads to lower price impacts (see e.g., Hasbrouck et al. 2022) by assuming that trading volume, V_i, to interval i takes the functional form $V_i = A \cdot (\Pi_{i,0})^\alpha$ for some constants $A \geqslant 0$ and $\alpha \in (0, 1)$. Using this functional form will only slightly modify our expression for equilibrium liquidity provision.

Note that we assume that all non-arbitrage trading corresponds to noise trading activity that nets out in terms of the price impact at the DEX. Therefore, changes in the DEX price only occur when innovations in public information (i.e., the price at the non-DEX venue) creates incentives for arbitrageurs to arbitrage the stale prices at the DEX.

3 Model Solution

Under the risk-neutral measure $\mathbb{Q}$, all assets must generate the same pay-off as a risk-free investment. Explicitly, letting $R^{\star}_{DEX,i}$ denote the equilibrium rate of return from investing in price interval i at the DEX, the following equation must hold for all i:

$$\mathbb{E}^{\mathbb{Q}}[R^{\star}_{DEX,i}] = e^{rT} \tag{14}$$

Then, applying Equations (6) and (13) to Equation (14) yields:

$$\mathbb{E}^{\mathbb{Q}}[R^{i}_{P\&L}] + \frac{\mathbb{E}^{\mathbb{Q}}[\Phi_i]}{\Pi^{\star}_{i,0}} = e^{rT} \tag{15}$$

where $\Pi^{\star}_{i,0}$ refers to the dollar value of equilibrium investment to price interval i. Note that both the expected ex-fee return from investing in price interval i, $R^{i}_{P\&L}$, and the expected total fees accrued in price interval i, Φ_i, are exogenous (see Equations 11 and 12). In turn, the endogenous equilibrium investment, $\Pi^{\star}_{i,0}$, can be derived directly from Equation (15). Furthermore, all other endogenous quantities can be determined from $\Pi^{\star}_{i,0}$. More formally, the following result provides an explicit solution for all endogenous quantities.

Proposition 3.1. *Equilibrium Model Solution*

The equilibrium investment, $\Pi^{\star}_{i,0}$, for each interval $i \in \mathbb{Z}$ is given explicitly as follows:

$$\Pi^{\star}_{i,0} = \frac{\mathbb{E}^{\mathbb{Q}}[\Phi_i]}{e^{rT} - \mathbb{E}^{\mathbb{Q}}[R^{i}_{P\&L}]} \tag{16}$$

where the $\mathbb{Q}$-measure total expected fee revenue for interval i, $\mathbb{E}^{\mathbb{Q}}[\Phi_i]$, is given explicitly as *follows:*

$$\mathbb{E}^{\mathbb{Q}}[\Phi_i] = \phi \times V \times \int_0^T \mathbb{Q}(P_t \in [\Psi_i, \Psi_{i+1}])\ dt \tag{17}$$

In turn, the equilibrium liquidity, $L_i^\star$, for each interval i is given explicitly as follows:

$$L_i^\star = \frac{\Pi_{i,0}^\star}{\gamma_i} \tag{18}$$

with γ_i is defined as follows:

$$\gamma_i = \left(\sqrt{\tilde{P}_{i,0}} - \sqrt{\Psi_i}\right) + \left(\frac{1}{\sqrt{\tilde{P}_{i,0}}} - \frac{1}{\sqrt{\Psi_{i+1}}}\right) \times P_0 \tag{19}$$

Finally, the equilibrium quantities of USDC, $USDC_{i,t}^\star$, and ETH, $ETH_{i,t}^\star$, in price interval i at time t are given as follows:

$$USDC_{i,t}^\star = \left(\sqrt{\tilde{P}_{i,t}} - \sqrt{\Psi_i}\right) \times L_i^\star, \qquad ETH_{i,t}^\star = \left(\frac{1}{\sqrt{\tilde{P}_{i,t}}} - \frac{1}{\sqrt{\Psi_{i+1}}}\right) \times L_i^\star \tag{20}$$

4 Equilibrium Results

We begin in Section 4.1 by clarifying the characteristics of the ex-fee return to liquidity providers, $R^i_{P\&L}$, for any given price interval i. In doing so, we demonstrate that providing liquidity to any price interval is equivalent to investing in a dynamic ETH-USDC portfolio with an ETH weight that declines in the ETH-USDC price. Subsequently, in Section 4.2, we compare ex-fee realized returns from providing liquidity at the DEX with those from direct investment in ETH or USDC. An important insight therein is that, in the absence of fees, providing liquidity to any price interval above the initial ETH-USDC price is dominated by holding ETH directly, whereas providing liquidity to any price interval below the initial

ETH-USDC price is dominated by holding USDC directly. These findings implicitly establish that, in equilibrium, the return from fees earned by providing liquidity to an interval must offset the opportunity cost of not investing directly in ETH or USDC instead. Finally, in Section 4.3, we provide our main results. More explicitly, we provide an approximate closed-form expression for equilibrium liquidity provision to any price interval. Moreover, we also demonstrate that the ex-fee return to liquidity provision is approximately equivalent to the return from a covered call trading strategy.

4.1 Ex-Fee Return to Liquidity Providers, $R^i_{P\&L}$

In the absence of trading fees, providing liquidity for ETH-USDC in price interval i is equivalent to investing in a portfolio of ETH and USDC except that the investor also faces a loss-versus-rebalancing (LVR) cost. A generalized version of this insight has been demonstrated previously for arbitrary AMMs (see Milionis et al. 2022); nonetheless, for completeness, we begin by re-establishing the result in our specific context of a CPAMM DEX with concentrated liquidity provision:

Proposition 4.1. *Liquidity Provision Is Investing in ETH-USDC Portfolio*

The instantaneous ex-fee return from providing liquidity to price interval i is given as follows:

$$\frac{d\Pi^\star_{i,t}}{\Pi^\star_{i,t}} = \omega^\star_{i,t}\frac{dP_t}{P_t} - \frac{l_{i,t}}{\Pi^\star_{i,t}}\ dt \tag{21}$$

where $\omega^\star_{i,t}$ denotes the equilibrium proportion of the inventory invested in ETH:

$$\omega^\star_{i,t} = \frac{ETH^\star_{i,t} \times P_t}{\Pi^\star_{i,t}} = \frac{ETH^\star_{i,t} \times P_t}{USDC^\star_{i,t} + ETH^\star_{i,t} \times P_t} \tag{22}$$

and $l_{i,t}$ denotes the instantaneous loss-versus-rebalancing (LVR), given explicitly as follows:

$$l_{i,t} = \begin{cases} \frac{L^\star_i \sigma_t^2 \sqrt{P_t}}{4} & \text{if } P_t \in [\Psi_i, \Psi_{i+1}] \\ 0 & \text{otherwise} \end{cases} \tag{23}$$

Explicitly, Equation (21) states that the instantaneous ex-fee liquidity provision return, $\frac{d\Pi^{\star}_{i,t}}{\Pi^{\star}_{i,t}}$, evolves akin to an ETH-USDC portfolio with an ETH portfolio weight that equals exactly the DEX inventory value of ETH as a proportion of the total DEX inventory value (i.e., $\omega^{\star}_{i,t} = \frac{ETH^{\star}_{i,t} \times P_t}{\Pi^{\star}_{i,t}}$). There is also a loss beyond the ETH-USDC portfolio instantaneous return, $\frac{l_{i,t}}{\Pi^{\star}_{i,t}}$, and this loss corresponds to the loss-versus-rebalancing (LVR) of Milionis et al. (2022); the LVR loss occurs because arbitrage trades occur at the DEX at stale prices and therefore impose losses on the liquidity providers.

To understand Proposition 4.1, it is important to recognize that liquidity providers for price interval i are pro-rata owners of the inventory associated with that price interval. Thus, the ex-fee return for providing liquidity for a price interval of ETH-USDC corresponds to the return from an ETH-USDC portfolio because the inventory for that price interval is a combination of ETH and USDC. In particular, as per Equation (21), an investor providing liquidity for price interval i experiences an instantaneous return, $\frac{d\Pi^{\star}_{i,t}}{\Pi^{\star}_{i,t}}$, proportional to instantaneous ETH returns, $\frac{dP_t}{P_t}$, exactly to the extent that ETH is weighted within the inventory for price interval i, $\omega^{\star}_{i,t}$.

Having shown that liquidity provision returns are akin to investment in an ETH-USDC portfolio, we next turn to clarifying the nature of that ETH-USDC portfolio in the context of a DEX with concentrated liquidity provision. To that end, we find that the ETH portfolio weighting, $\omega^{\star}_{i,t}$, is dynamic and more specifically that it depends negatively on the ETH-USDC price, P_t. More explicitly, we establish the following result:

Proposition 4.2. *ETH Portfolio Weight Declines Monotonically From Unity to Zero*
When the ETH-USDC price level is below the price interval (i.e., $P_t < \Psi_i$), then the liquidity portfolio is equivalent to holding ETH directly:

$$P_t < \Psi_i \implies \omega^{\star}_{i,t} = 1 \tag{24}$$

When the ETH-USDC price level is within the price interval (i.e., $P_t \in [\Psi_i, \Psi_{i+1}]$), then the liquidity portfolio is equivalent to holding an ETH-USDC portfolio with dynamic weighting:

$$P_t \in [\Psi_i, \Psi_{i+1}] \implies \omega^\star_{i,t} = \omega^\star_i(P_t) \tag{25}$$

where $\omega^\star_i : [\Psi_i, \Psi_{i+1}] \mapsto [0, 1]$ is a continuous and monotonically decreasing function that satisfies $\omega^\star_i(\Psi_i) = 1$ and $\omega^\star_i(\Psi_{i+1}) = 0$.

When the ETH-USDC price level is above the price interval (i.e., $P_t > \Psi_{i+1}$), then the liquidity portfolio is equivalent to holding USDC directly:

$$P_t > \Psi_{i+1} \implies \omega^\star_{i,t} = 0 \tag{26}$$

Proposition 4.2 establishes not only that the ETH portfolio weight, $\omega^\star_{i,t}$, is dynamic but also that the ETH portfolio weight specifically evolves as a decreasing function of the ETH-USDC price, P_t. In more detail, the ETH portfolio weight is unity when the ETH-USDC price is fully below the interval but then declines continuously to zero as the ETH-USDC price moves through the interval and finally remains zero thereafter when the ETH-USDC price is fully above the interval. It is noteworthy that this result would not hold if we were to assume uniform liquidity provision; more explicitly, under uniform liquidity provision and a CPAMM (e.g., Uniswap V1 and V2), $\omega^\star_{i,t}$ is constant and does not depend on P_t (see, e.g., Angeris et al. 2021).[5]

To understand Proposition 4.2, it is important to recognize that increases in the ETH-USDC price lead to net buying of ETH at the DEX and that net buying of ETH at the DEX reduces the proportion of ETH inventory held by the DEX. To provide more context,

[5] To provide additional context, a CPAMM is a special case within a broader class of AMMs known as Geometric Mean Market Makers (G3Ms). Notably, under uniform liquidity provision, all G3Ms possess static portfolio weights (see, e.g., Angeris and Chitra 2020 and Evans 2021). Proposition 4.2 demonstrates that this static portfolio weights result does not hold under concentrated liquidity thereby highlighting an important difference between uniform liquidity and concentrated liquidity.

recall that the DEX employs a mechanical pricing function (i.e., CPAMM) and that this mechanical pricing function does not directly incorporate innovations in public information. Thus, when a positive innovation generates an increase in the ETH-USDC price away from the DEX, an arbitrage opportunity arises whereby traders can momentarily buy at stale ETH-USDC prices from the DEX and sell at the new higher ETH-USDC prices away from the DEX.[6] In turn, this arbitrage trading represents net ETH buying at the DEX which reduces the ETH weight in the liquidity portfolio. More concretely, the reduction in the ETH portfolio weight arises from the ETH buy trade because a trade that buys ETH at the DEX is implemented as an ETH-USDC swap where the trader receives ETH inventory from the DEX in return for providing payment as USDC inventory to the DEX, thereby lowering the liquidity portfolio ETH weight.

4.2 Comparative Insights Regarding LP Returns

We next turn to examining how providing liquidity at a DEX with concentrated liquidity provision compares to investing directly in ETH or USDC. Our first result in that regard, Proposition 4.3, highlights that, in the absence of fees, providing liquidity at a DEX with concentrated liquidity provision is always dominated either by investing directly in ETH or by investing directly in USDC:

Proposition 4.3. *Ex-Fee LP Return Is Dominated by ETH or USDC*
For any price interval i that is above the initial price level (i.e., $\Psi_i \geqslant P_0$), the ex-fee realized return is lower than then realized return from investing in ETH directly:

$$R^i_{P\&L} \leqslant \frac{P_T}{P_0} =: R^{ETH} \tag{27}$$

Moreover, for any price interval i that is below the initial price level (i.e., $\Psi_{i+1} \leqslant P_0$), the ex-fee realized return is lower than the realized return from investing in USDC directly:

[6] This stale price arbitrage trading was first studied by Capponi and Jia (2021).

$$R^i_{P\&L} \leqslant 1 =: R^{USDC} \tag{28}$$

Moreover, $\mathbb{E}[R^i_{P\&L}] < \mathbb{E}[R^{ETH}]$ *for any price interval* i*.*

To provide more detail, Proposition 4.3 establishes that the realized return from investing in ETH always exceeds the ex-fee realized return from liquidity provision to any price interval above the initial ETH-USDC price (Equation 27), and the realized return from holding USDC always exceeds the ex-fee realized return from liquidity provision to any price interval below the initial ETH-USDC price (Equation 28). In order to explain these results, we first show (via Propositions 4.4 and 4.5) that the returns on the liquidity portfolio for a given price interval are higher for price intervals further away from the initial ETH-USDC price. Then, Proposition 4.3 is derived from the fact that investing in a price interval infinitely far above the current ETH-USDC price is equivalent to investing directly in ETH while investing in a price interval infinitely far below the current ETH-USDC price is equivalent to holding USDC.

Proposition 4.4. *Ex-Fee LP Return Increases Above The Money, Converges to ETH Return* *For any two price intervals* $i > j$ *such that the smaller price interval is above the initial price level (i.e.,* $\Psi_j \geqslant P_0$*), then the ex-fee realized return of the higher price interval exceeds that of the lower price interval:*

$$R^i_{P\&L} \geqslant R^j_{P\&L} \tag{29}$$

Moreover, the ex-fee realized return for providing liquidity to a price interval i *converges to the return from investing directly in ETH as the interval becomes arbitrarily far above the initial price level (i.e.,* $i \to \infty$*):*

$$\lim_{i\to\infty} R^i_{P\&L} = \frac{P_T}{P_0} = R^{ETH} \tag{30}$$

Proposition 4.4 finds that, when comparing two price intervals $i > j$ above the initial ETH-USDC price (i.e., $\Psi_j \geqslant P_0$), then the realized ex-fee return for providing liquidity to the higher interval always exceeds that for providing liquidity to the lower interval (i.e., $R^i_{P\&L} \geqslant R^j_{P\&L}$). Proposition 4.4 also finds that the ex-fee realized return for providing liquidity to price intervals above the initial ETH-USDC price converges to the ETH return as the interval being examined becomes infinitely far above the initial ETH-USDC price (i.e., $\lim_{i\to\infty} R^i_{P\&L} = R^{ETH}$).

The first part of Proposition 4.4 arises because the ETH portfolio weights for any pair of price intervals above the initial ETH-USDC price can differ only when the ETH-USDC price is above its initial level. Moreover, a higher price interval possesses a higher ETH portfolio weight in general and thus generates a higher realized return because it incurs a larger gain whenever the ETH price increases. To provide more depth, Proposition 4.2 establishes that both price intervals possess ETH portfolio weights of unity when the ETH-USDC price is below the intervals and ETH portfolio weights of zero when the ETH-USDC price is above the intervals. In turn, the ETH portfolio weights for the pair of price intervals differ only when the ETH-USDC price is above the lower bound of the smaller interval but below the upper bound of the larger interval. Since Proposition 4.4 examines only price intervals above the initial ETH-USDC price, the ETH portfolio weights for such intervals differ only in cases that the ETH-USDC price increases due to the fact that entering either price interval requires an ETH-USDC price increase. Then, because the higher price interval possesses a higher ETH portfolio weight always, the higher price interval therefore always realizes a higher return. Intuitively, the higher price interval is more exposed to ETH when ETH-USDC prices increase and thus it incurs larger gains and realizes a higher return.

The second part of Proposition 4.4 arises because the ETH portfolio weight is unity when the ETH-USDC price is below the price interval. Thus, an ETH-USDC portfolio with a weight of unity on ETH is a portfolio of only ETH and therefore equivalent to holding ETH directly. Then, since the ETH-USDC price never reaches a price interval infinitely far above the current price level, that limiting price interval possesses an ETH portfolio weight of unity with probability one and consequently providing liquidity to that interval is equivalent to holding ETH directly.

Proposition 4.5. <u>*Ex-Fee LP Return Decreases Below The Money, Converges to USDC Return*</u> *For any two price intervals $i > j$ such that the larger price interval is fully below the initial price level (i.e., $\Psi_{i+1} \leqslant P_0$), then the ex-fee realized return of the lower price interval exceeds that of the higher price interval:*

$$R^{j}_{P\&L} \geqslant R^{i}_{P\&L} \tag{31}$$

Moreover, the ex-fee realized return for providing liquidity to a price interval i converges to the return from holding USDC directly as the interval becomes arbitrarily far below the initial price level (i.e., $i \to -\infty$):

$$\lim_{i \to -\infty} R^{i}_{P\&L} = 1 = R^{USDC} \tag{32}$$

As a consequence, $\mathbb{E}[R^{i}_{P\&L}] < \mathbb{E}[R^{USDC}]$ for any price interval i.

Proposition 4.5 finds that, when comparing two price intervals $i > j$ below the initial ETH-USDC price (i.e., $\Psi_{i+1} \leqslant P_0$), then the realized ex-fee return for providing liquidity to the lower interval always exceeds that for providing liquidity to the higher interval (i.e., $R^{j}_{P\&L} \geqslant R^{i}_{P\&L}$). Proposition 4.5 also finds that the ex-fee realized return for providing liquidity to price intervals below the initial ETH-USDC price converges to the USDC return as the interval being examined becomes infinitely far below the initial ETH-USDC price (i.e., $\lim_{i \to -\infty} R^{i}_{P\&L} = R^{USDC}$).

Similar to the intuition for Proposition 4.4, the first part of Proposition 4.5 arises because the ETH portfolio weights for any pair of price intervals below the initial ETH-USDC price differ only when the ETH-USDC price is below its initial level; additionally, the lower price interval possesses a lower ETH portfolio weight and thus the lower price interval incurs a higher realized return because it incurs a smaller loss from an ETH price decrease. To provide more depth, Proposition 4.2 establishes that both price intervals possess ETH portfolio weights of unity when the ETH-USDC price is below the intervals and ETH portfolio weights of zero when the ETH-USDC price is above the intervals. As a consequence, the ETH portfolio weights for the pair of price intervals differ only when the ETH-USDC price is below the upper bound of the higher interval but above the lower bound of the lower interval. Since Proposition 4.5 examines only price intervals below the initial ETH-USDC price, the ETH portfolio weights for such intervals differ only in cases that the ETH-USDC price decreases due to the fact that entering either price interval requires an ETH-USDC price decrease. Then, because the lower price interval possesses a lower ETH portfolio weight always (see Proposition 4.2), the lower price interval therefore always realizes a higher return. Intuitively, the lower price interval is less exposed to ETH when ETH-USDC prices decline and thus it faces smaller losses and realizes a higher return.

The second part of Proposition 4.5 arises because the ETH portfolio weight is zero when the ETH-USDC price is above the price interval (see Proposition 4.2) and also because the ETH-USDC price at liquidation is above the price interval infinitely far below with probability one. Notably, an ETH-USDC portfolio with a weight of zero on ETH is a portfolio of only USDC and thus equivalent to holding USDC directly. In turn, since the ETH-USDC price never reaches a price interval that is infinitely far below the initial ETH-USDC price, then the price interval infinitely far below possesses an ETH portfolio weight of zero with probability one, implying that providing liquidity to that interval is equivalent to holding USDC directly.

4.3 Equilibrium Liquidity Provision, $\Pi^{\star}_{i,0}$

Our final set of results concerns equilibrium liquidity provision, $\Pi^{\star}_{i,0}$. We begin by highlighting that, in the absence of fees (i.e., $\phi = 0$), the equilibrium liquidity provision is zero for all price intervals:

Corollary 4.6. *The Need For Fees at a DEX*

If trading fees are zero (e.g., $\phi = 0$), then liquidity provision is also zero:

$$\phi = 0 \implies \Pi^{\star}_{i,0} = 0 \text{ for all } i \tag{33}$$

Corollary 4.6 arises due to Proposition 4.3. In particular, since ex-fee realized returns from liquidity provision are below realized returns for at least one other investment opportunity (i.e., direct holding of ETH or USDC), DEX liquidity provision is therefore a dominated strategy in the absence of fees. In turn, in the absence of fees, no investor would provide liquidity to the DEX in equilibrium and there is a need for fees just as in the case of a DEX with uniform liquidity provision (see, e.g., Hasbrouck et al. 2022).

Nonetheless, fees are non-zero in practice and Equation (16) characterizes equilibrium liquidity provision for general fee levels. Unfortunately, Equation (16) is somewhat opaque in general, so we offer an intuitive and easy to use approximate expression for liquidity provision in our next result:

Proposition 4.7. *Equilibrium Liquidity Provision as $\Delta \to 0^+$*

Given a fixed price level P, we consider an arbitrary sequence of price intervals, $\{i(\Delta_n, P)\}_{n\in\mathbb{N}}$, such that $\lim_{n\to\infty} \Delta_n = 0$ where each price interval is selected to contain P for each Δ (i.e., $P \in [\Psi_{i(\Delta,P)}, \Psi_{i(\Delta,P)+1}]$). In turn, we apply Proposition 3.1 to construct the associated liquidity provision sequence, $\{\Pi^{\star}_{i(\Delta_n,P)}\}_{n\in\mathbb{N}}$, and we thereby derive the limiting liquidity provision, $\Pi^{\star}(P)$, for each price level, P, as follows:

$$\Pi^{\star}(P) := \lim_{\Delta\to 0^+} \frac{1}{\Delta}\Pi^{\star}_{i(\Delta,P),0} = \begin{cases} \dfrac{\phi\times V\times P_0\times \int_0^T f(p,t)\ dt}{e^{rT}\mathcal{C}(P,T)} & \text{if } P \geqslant P_0 \\ \dfrac{\phi\times V\times P\times \int_0^T f(p,t)\ dt}{e^{rT}\mathcal{C}(P,T) - e^{rT}(P_0 - P)} & \text{if } P < P_0 \end{cases} \tag{34}$$

where $f(p,t)$ denotes the $\mathbb{Q}$-measure density of $p_t := \log(P_t)$ and $\mathcal{C}(K,\tau) := e^{-rT}\mathbb{E}^{\mathbb{Q}}[(P_\tau - K)^+]$ refers to the price for a European call option with ETH-USDC as the underlying, K

as the strike price and τ *as the time to maturity.*

As a corollary, when the ETH-USDC price follows geometric Brownian motion (i.e., $\sigma_t = \sigma > 0$*), then the hypothesis of* $f(p,t)$ *being continuous is met and Equation (34) simplifies due to the following well-known results:*

$$f(p,t) = f_{\mathcal{N}}(p; p_0 + (r - \frac{\sigma^2}{2})t, \sigma^2 t), \qquad \mathcal{C}(K,\tau) = P_0 \cdot F_{\mathcal{N}}(d_+) - K \cdot e^{-r\tau} \cdot F_{\mathcal{N}}(d_-) \quad (35)$$

where $f_{\mathcal{N}}(\cdot; \mu, v)$ *refers to the density of a normal random variable with mean* μ *and variance* v*,* $F_{\mathcal{N}}$ *denotes the cumulative distribution function for a standard normal random variable and* $d_\pm$ *is given explicitly as follows:*

$$d_\pm = \frac{1}{\sigma\sqrt{\tau}}\Big(\log\big(\frac{P_0}{K}\big) + \big(r \pm \frac{\sigma^2}{2}\big)\tau\Big) \quad (36)$$

Notably, Δ is typically small in practice so that a simplified practical expression for equilibrium liquidity provision can be deduced by considering the limiting case $\Delta \to 0^+$. In particular, Proposition 4.7 can be applied to yield the following approximate value for equilibrium liquidity provision for an arbitrary price interval i:

$$\Pi^\star_{i,0} \approx \Pi^\star\Big(P(i)\Big) \times \Delta \quad (37)$$

where $\Pi^\star$ is given explicitly by Equation (34) and an arbitrarily chosen $P(i) \in [\Psi_i, \Psi_{i+1}]$. We suggest letting $P(i)$ correspond to the geometric average of the interval as follows:

$$P(i) := \sqrt{\Psi_i \Psi_{i+1}} \quad (38)$$

To provide more detail, Proposition 4.7 fixes a price level, $P \in \mathbb{R}_+$, and considers the equilibrium liquidity provision, $\Pi^\star_{i_n,0} := \Pi^\star_{i(\Delta_n,P),0}$, for the price interval $i_n := i(\Delta_n, P)$ containing the price P when the DEX's price grid is defined by an arbitrary $\Delta_n > 0$. We then construct an arbitrary sequence of $\{\Delta_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \Delta_n = 0$ and generate the corresponding equilibrium liquidity provision values, $\{\Pi^{\star}_{i(\Delta_n,P),0}\}_{n\in\mathbb{N}}$ where each equilibrium liquidity provision value is determined as per our equilibrium solution Equation (16). Equation (34) provides the limit of such an arbitrary sequence (with the appropriate scaling by $\frac{1}{\Delta}$). Since Δ is small in practice, we suggest using the well-defined limit point $\Pi^{\star}(P)$, after reversing the scaling (i.e., multiply by Δ), to proxy for liquidity provision in practice. This suggested process is implemented explicitly through Equations (37) and (38).

As a final point, the limiting case when the size of the price intervals vanish (i.e., $\Delta \to 0^+$) is also useful to highlight the key economic channel driving liquidity provision returns at a DEX with concentrated liquidity. To that end, recall that Proposition 4.2 establishes that providing liquidity to the DEX entails holding exclusively ETH below the price interval to which liquidity is provided and holding exclusively USDC above that price interval. Then, as $\Delta \to 0^+$, each DEX price interval collapses to a single price level and thus the investment profile from DEX liquidity provision becomes equivalent to that of holding ETH up to the price level at which liquidity is provided and holding USDC beyond that price level. More formally, the return from liquidity provision to the DEX for *any* price level is equivalent to that from investing in ETH and selling an ETH-USDC European call option where the call option possesses a strike equal to the price level at which liquidity is being provided and is sold at its intrinsic value rather than its market value:

Proposition 4.8. *Consider an arbitrary sequence of price intervals, $\{i(\Delta_n, P)\}_{n\in\mathbb{N}}$, such that $\lim_{n\to\infty} \Delta_n = 0$ where each price interval is selected to contain P for each Δ (i.e., $P \in [\Psi_{i(\Delta,P)}, \Psi_{i(\Delta,P)+1}]$). Then, the ex-fee realized return for DEX liquidity provision converges to that from holding ETH and selling an ETH-USDC European call option with a strike equal to P if the call option is sold at its intrinsic value:*

$$\lim_{\Delta\to 0^+} R^{i(\Delta,P)}_{P\&L} = \frac{P_T - \mathcal{C}_I(P_T, P)}{P_0 - \mathcal{C}_I(P_0, P)} \tag{39}$$

where $\mathcal{C}_I(P_t, K) := (P_t - K)^+$ denotes the intrinsic value of the call option where the intrin-

sic value is defined as the payout from exercising an otherwise equivalent American option immediately.

Note that the numerator of (39), $P_T - \mathcal{C}_I(P_T, P)$, is equal to the pay-off from holding ETH and shorting an ETH-USDC call option against that ETH position (i.e., a covered call). In particular, the call option is characterized by the strike price P and the maturity T where P represents the price level at which liquidity is being provided and T the investor's investment horizon. To provide more context, a liquidity provider receives the ETH price, P_T, if the ETH-USDC price ends below the level at which liquidity is provided (i.e., $P_T < P$). In that case, the call option expires worthless (i.e., $\mathcal{C}_I(P_T, P) = (P_T - P)^+ = 0$ when $P_T < P$) and thus the liquidity provider holds only ETH as per a covered call strategy. In the alternative case that the terminal ETH-USDC price ends above the level at which liquidity is provided (i.e., $P_T > P$) then the liquidity provider receives a pay-off equal to the strike price of P. In this case, the call option is optimally exercised at strike price P and therefore the investor is forced to sell her ETH in return for P units of USDC thereby generating the same pay-off as a covered call strategy.[7]

5 Conclusion

We study optimal DEX liquidity provision when the DEX allows investors to concentrate liquidity to pre-specified price intervals (e.g., Uniswap V3). Importantly, and in contrast to a limit order book, providing concentrated liquidity to a DEX entails providing *two-way* liquidity so that whenever an investor's liquidity is utilized for an exchange, the investor automatically becomes a liquidity provider of the asset for which their liquidity was exchanged. For this reason, providing liquidity for an ETH-USDC exchange entails investing in a port-

[7]Note that the numerator of Equation (39) matches a covered call strategy pay-off but the denominator in Equation (39) does not match the cost of initiating a covered call strategy. The reason for this is that, as $\Delta \to 0^+$, the DEX replicates the call option pay-off in a fashion similar to the "stop-loss start-gain" strategy of Seidenverg (1988). Importantly, this strategy of replicating a call option does not entail paying the market value of the call upfront but rather entails paying only the intrinsic value upfront and making additional payments thereafter because the strategy is not self-financing (see Carr and Jarrow 1990 for details).

folio of ETH and USDC with dynamic weights that evolve with the underlying ETH-USDC price. This feature of DEXs with concentrated liquidity generates new trade-offs faced by liquidity providers and therefore characterizes the level of expected fee revenue necessary to incentivize liquidity provision to a particular price interval. In particular, we show that without fees, providing liquidity to a particular price interval is always dominated by directly investing in either ETH or USDC. Thus, for any given level of fee revenue, liquidity provision will adjust so that the pro-rata return from fees paid to that price interval offsets the opportunity cost of investing in other assets (e.g., ETH or USDC). In turn, we characterize the equilibrium liquidity provision and provide a simple approximate expression that can be useful for empirical work and comparative statics given the opaque nature of the expression for equilibrium liquidity provision in the general case.

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Appendices

A Proofs

A.1 Proof of Proposition 3.1

The $\mathbb{Q}$-measure is such that all investments must generate the same expected return as the risk-free investment (i.e., Equation 14 must hold). Then, applying Equations (6) and (13) to Equation (14) yields Equation (15). Solving for $\Pi^{\star}_{i,0}$ in Equation (15) then yields Equation (16).

Equation (17) follows from applying the $\mathbb{Q}$-expectation to Equation (12) and then applying Tonelli's Theorem to interchange the expectation and the integral.

In order to derive Equations (18) and (19), note that the equilibrium value $L^{\star}_i$ does not depend on time t. Therefore, setting $t = 0$ and applying Equation (9) to Equation (8) then implies that, in equilibrium,

$$\Pi^{\star}_{i,0} = L^{\star}_i \left((\sqrt{\tilde{P}_{i,0}} - \sqrt{\Psi_i}) + (\frac{1}{\sqrt{\tilde{P}_{i,0}}} - \frac{1}{\sqrt{\Psi_{i+1}}}) \times P_0 \right)$$

which after rearranging gives our expression for $L^{\star}_i$.

Finally, Equation (20) follows from applying the previous equilibrium solutions to Equation (9).

A.2 Proof of Proposition 4.1

Applying the equilibrium solutions from Proposition 3.1 to Equation (8) implies that equilibrium liquidity for price interval i at time t, $\Pi^{\star}_{i,t}$, can be written as a univariate function, $\Pi^{\star}_i$, of the time t price P_t as follows:

$$\Pi^{\star}_{i,t} = \Pi^{\star}_i(P_t) := USDC^{\star}_i(P_t) + ETH^{\star}_i(P_t) \times P_t \tag{A.1}$$

with $USDC_i^\star(P_t)$ and $ETH_i^\star(P_t)$ denoting the equilibrium USDC and ETH holdings, written explicitly as a function of the time t price P_t as follows:

$$USDC_i^\star(P_t) := \left(\sqrt{\tilde{P}_{i,t}} - \sqrt{\Psi_i}\right) \times L_i, \qquad ETH_i^\star(P_t) := \left(\frac{1}{\sqrt{\tilde{P}_{i,t}}} - \frac{1}{\sqrt{\Psi_{i+1}}}\right) \times L_i \tag{A.2}$$

where $\tilde{P}_{i,t}$ denotes the projection of P_t onto $[\Psi_i, \Psi_{i+1}]$ as per Equation (10).

Note that $\Pi_i^\star(P_t)$ is continuously differentiable everywhere but not twice continuously differentiable everywhere. In particular, $\Pi_i^\star(P_t)$ is not twice continuously differentiable at $P_t = \Psi_i$ and $P_t = \Psi_{i+1}$ even though it is twice continuously differentiable at all other points. Then, since the hypothesis for the standard Ito's lemma is not satisfied, we instead invoke a generalized version of Ito's lemma for functions twice continuously differentiable at all but finitely many points (see Chapter 3.6 Section D of Karatzas and Shreve 1991):

$$d\Pi_{i,t}^\star = d\Pi_i^\star(P_t) = \frac{d\Pi_i^\star}{dP_t} dP_t + \frac{1}{2}\frac{d^2\Pi_i^\star}{dP_t^2} d[P_t, P_t] \tag{A.3}$$

Notably, although we require a generalized version of Ito's lemma, this generalized version reduces to the form of the usual Ito's lemma because $\Pi_i^\star$ is continuously differentiable even at the two points at which the second derivative does not exist. Then, to proceed, we compute the first derivative of $\Pi_i^\star$, $\frac{d\Pi_i^\star}{dP_t}$ explicitly as follows:

$$\frac{d\Pi_i^\star}{dP_t} = ETH_i^\star(P_t) \tag{A.4}$$

Furthermore, we compute the second derivative of $\Pi_i^\star$, $\frac{d^2\Pi_i^\star}{dP_t^2}$, at all points where this second derivative exists (i.e., when $P_t \neq \Psi_i, \Psi_{i+1}$):

$$\frac{d^2\Pi_i^\star}{dP_t^2} = \begin{cases} 0 & \text{if } P_t \notin [\Psi_i, \Psi_{i+1}] \\ -\frac{L_i^\star}{2\sqrt{P_t^3}} & \text{if } P_t \in (\Psi_i, \Psi_{i+1}) \end{cases} \tag{A.5}$$

Then, Equations (A.4) - (A.5) imply that for $P_t \notin [\Psi_i, \Psi_{i+1}]$, Equation (A.3) becomes:

$$d\Pi^\star_{i,t} = ETH^\star_i(P_t)\ dP_t = ETH^\star_i(P_t) \times P_t \frac{dP_t}{P_t} \tag{A.6}$$

which, in turn, implies:

$$\frac{d\Pi^\star_{i,t}}{\Pi^\star_{i,t}} = \frac{ETH^\star_i(P_t) \times P_t}{\Pi^\star_{i,t}} \frac{dP_t}{P_t} = \omega^\star_{i,t} \frac{dP_t}{P_t} \tag{A.7}$$

which establishes Proposition 4.1 for $P_t \notin [\Psi_i, \Psi_{i+1}]$.

Similarly, Equations (A.4) - (A.5) imply that for $P_t \in (\Psi_i, \Psi_{i+1})$ Equation (A.3) becomes:

$$d\Pi^\star_{i,t} = ETH^\star_i(P_t)\ dP_t - \frac{L^\star_i}{4\sqrt{P_t^3}} d[P_t, P_t] = P_t \times ETH^\star_i(P_t)\ \frac{dP_t}{P_t} - \frac{L^\star_i \sigma_t^2 \sqrt{P_t}}{4} dt \tag{A.8}$$

which, in turn, implies:

$$\frac{d\Pi^\star_{i,t}}{\Pi^\star_{i,t}} = \frac{P_t \times ETH^\star_i(P_t)}{\Pi^\star_{i,t}} \frac{dP_t}{P_t} - \frac{L^\star_i \sigma_t^2 \sqrt{P_t}}{4\Pi^\star_{i,t}} dt = \omega^\star_{i,t} \frac{dP_t}{P_t} - \frac{l_{i,t}}{\Pi^\star_{i,t}} dt \tag{A.9}$$

thereby completing the proof. □

A.3 Proof of Proposition 4.2

When $P_t < \Psi_i$, Equation (20) implies $USDC^\star_{i,t} = 0$ and thus $P_t < \Psi_i \implies \omega^\star_{i,t} = 1$, thereby establishing Equation (24).

When $P_t > \Psi_{i+1}$, Equation (20) implies $ETH^\star_{i,t} = 0$ and thus $P_t > \Psi_{i+1} \implies \omega^\star_{i,t} = 0$, thereby establishing Equation (26).

For $P_t \in [\Psi_i, \Psi_{i+1}]$, Equation (20) implies that $USDC^\star_{i,t} = USDC^\star_i(P) := \left(\sqrt{P} - \sqrt{\Psi_i}\right) \times L^\star_i$ and $ETH^\star_{i,t} = ETH^\star_i(P) := \left(\frac{1}{\sqrt{P}} - \frac{1}{\sqrt{\Psi_{i+1}}}\right) \times L^\star_i$ which, in turn, establishes Equation (25) whereby

$$\omega_{i,t}^{\star} = \omega_i^{\star}(P) := \frac{ETH_i^{\star}(P) \times P}{USDC_i^{\star}(P) + ETH_i^{\star}(P) \times P} \tag{A.10}$$

By direct verification, $\omega_i^{\star}(\Psi_i) = 1$, $\omega_i^{\star}(\Psi_{i+1}) = 0$ and moreover $\omega_i^{\star} : [\Psi_i, \Psi_{i+1}]$ is continuous. To conclude the proof, we show that $\omega_i^{\star}$ is monotonically decreasing. To establish that result, we first note that $\frac{d\omega_i^{\star}(P)}{dP} \leqslant 0$ if and only if

$$\frac{d}{dP}[ETH_i^{\star}(P) \times P] \times USDC_i^{\star}(P) - \frac{dUSDC_i^{\star}(P)}{dP} ETH_i^{\star}(P) \times P \leqslant 0$$

which after substituting the expression from Equation (20) and rearranging holds if and only if

$$2\sqrt{\frac{\Psi_i}{\Psi_{i+1}}} \leqslant \sqrt{\frac{\Psi_i}{P}} + \sqrt{\frac{P}{\Psi_{i+1}}} \tag{A.11}$$

Finally, in order to show that (A.11) always holds, note that $P \in [\Psi_i, \Psi_{i+1}]$ implies $\frac{\Psi_i}{P}, \frac{P}{\Psi_{i+1}} \in [0, 1]$. In turn, $\frac{\Psi_i}{P} \leqslant \sqrt{\frac{\Psi_i}{P}}$ and $\frac{P}{\Psi_{i+1}} \leqslant \sqrt{\frac{P}{\Psi_{i+1}}}$ and thus $\frac{\Psi_i}{P} + \frac{P}{\Psi_{i+1}} \leqslant \sqrt{\frac{\Psi_i}{P}} + \sqrt{\frac{P}{\Psi_{i+1}}}$ so that $2\sqrt{\frac{\Psi_i}{\Psi_{i+1}}} \leqslant \frac{\Psi_i}{P} + \frac{P}{\Psi_{i+1}}$ is a sufficient condition for inequality (A.11) to hold. Finally, direct verification yields:

$$2\sqrt{\frac{\Psi_i}{\Psi_{i+1}}} \leqslant \frac{\Psi_i}{P} + \frac{P}{\Psi_{i+1}} \Leftrightarrow \left(\sqrt{\frac{\Psi_i}{P}} - \sqrt{\frac{P}{\Psi_{i+1}}}\right)^2 \geqslant 0 \tag{A.12}$$

which completes the proof. □

A.4 Lemma A.1

Lemma A.1. *Price Interval Above Initial Price Level*

For any price interval i such that $\Psi_i > P_0$, the following results hold:

(a) For $P_T \leqslant \Psi_i$, $R^i_{P\&L} = \frac{P_T}{P_0}$

(b) For $P_T \geqslant \Psi_{i+1}$, $R^i_{P\&L} = \frac{\Psi_i}{P_0}\sqrt{1+\Delta}$

(c) For $P_T \in [\Psi_i, \Psi_{i+1}]$, $\frac{\Psi_i}{P_0} \leqslant R^i_{P\&L} \leqslant \frac{P_T}{P_0}$

Proof.

First note that whenever $\Psi_i > P_0$ then $\tilde{P}_{i,0} = \Psi_i$ and therefore, after substituting into Equation (11), $R^i_{P\&L}$ is given explicitly as:

$$R^i_{P\&L} = \frac{\left(\sqrt{\tilde{P}_{i,T}} - \sqrt{\Psi_i}\right) + \left(\frac{1}{\sqrt{\tilde{P}_{i,T}}} - \frac{1}{\sqrt{\Psi_{i+1}}}\right) \times P_T}{\left(\frac{1}{\sqrt{\Psi_i}} - \frac{1}{\sqrt{\Psi_{i+1}}}\right) \times P_0} \tag{A.13}$$

Then, $P_T \leqslant \Psi_i$ implies $\tilde{P}_{i,T} = \Psi_i$ which, when applied to Equation (A.13), yields (a). Similarly, $P_T \geqslant \Psi_{i+1}$ implies $\tilde{P}_{i,T} = \Psi_{i+1}$ which, when applied to Equation (A.13), yields (b).

To establish (c), note that when $P_T \in [\Psi_i, \Psi_{i+1}]$, then $\tilde{P}_{i,T} = P_T$ and thus $R^i_{P\&L} = \widehat{R}^i_{P\&L}(P_T)$ with the latter being given explicitly as follows:

$$\widehat{R}^i_{P\&L}(P_T) = \frac{\left(\sqrt{P_T} - \sqrt{\Psi_i}\right) + \left(\frac{1}{\sqrt{P_T}} - \frac{1}{\sqrt{\Psi_{i+1}}}\right) \times P_T}{\left(\frac{1}{\sqrt{\Psi_i}} - \frac{1}{\sqrt{\Psi_{i+1}}}\right) \times P_0} \tag{A.14}$$

Further, $\widehat{R}^i_{P\&L}$ is differentiable with:

$$\frac{d\widehat{R}^i_{P\&L}}{dP} = \frac{\frac{1}{\sqrt{P}} - \frac{1}{\sqrt{\Psi_{i+1}}}}{\frac{1}{\sqrt{\Psi_i}} - \frac{1}{\sqrt{\Psi_{i+1}}}} \times \frac{1}{P_0} \tag{A.15}$$

and thus $P_T \in [\Psi_i, \Psi_{i+1}]$ implies:

$$\frac{d\widehat{R}^i_{P\&L}}{dP} \in [0, \frac{1}{P_0}] \tag{A.16}$$

Further, $\widehat{R}^i_{P\&L}(P_T)$ is continuous and differentiable over the interval $[\Psi_i, \Psi_{i+1}]$ and therefore by the mean value theorem, for any $P_T \in (\Psi_i, \Psi_{i+1}]$ there exists $P' \in (\Psi_i, \Psi_{i+1}]$ such that

$$\widehat{R}^i_{P\&L}(P_T) - \widehat{R}^i_{P\&L}(\Psi_i) = \frac{d\widehat{R}^i_{P\&L}}{dP}(P')(P_T - \Psi_i) \leqslant \frac{1}{P_0}(P_T - \Psi_i)$$

where the last inequality comes from the fact that $\frac{d\widehat{R}^i_{P\&L}}{dP} < \frac{1}{P_0}$. In addition, $\frac{d\widehat{R}^i_{P\&L}}{dP} \geqslant 0$ implies that $\widehat{R}^i_{P\&L}(P_T) \geqslant \widehat{R}^i_{P\&L}(\Psi_i) = \frac{\Psi_i}{P_0}$ and therefore $\frac{\Psi_i}{P_T} \leqslant R^i_{P\&L} = \left(\widehat{R}^i_{P\&L}(P_T) - \widehat{R}^i_{P\&L}(\Psi_i)\right) + \widehat{R}^i_{P\&L}(\Psi_i) \leqslant \frac{P_T}{P_0}$ as desired for (c).

□

A.5 Lemma A.2

Lemma A.2. *Price Interval Below Initial Price Level*

For any price interval i such that $\Psi_{i+1} < P_0$, the following results hold:

(a) For $P_T \leqslant \Psi_i$, $R^i_{P\&L} = \frac{P_T}{\Psi_i\sqrt{1+\Delta}}$

(b) For $P_T \geqslant \Psi_{i+1}$, $R^i_{P\&L} = 1$

(c) For $P_T \in [\Psi_i, \Psi_{i+1}]$, $\frac{1}{\sqrt{1+\Delta}} \leqslant R^i_{P\&L} \leqslant 1$

Proof.

First note that $\Psi_{i+1} < P_0$ and Equation (11) imply that the $R^i_{P\&L}$ is given explicitly as follows:

$$R^i_{P\&L} = \frac{\left(\sqrt{\tilde{P}_{i,T}} - \sqrt{\Psi_i}\right) + \left(\frac{1}{\sqrt{\tilde{P}_{i,T}}} - \frac{1}{\sqrt{\Psi_{i+1}}}\right) \times P_T}{\sqrt{\Psi_{i+1}} - \sqrt{\Psi_i}} \tag{A.17}$$

Then, $P_T \leqslant \Psi_i$ implies $\tilde{P}_{i,T} = \Psi_i$ which, when applied to Equation (A.17), yields (a). Similarly, $P_T \geqslant \Psi_{i+1}$ implies $\tilde{P}_{i,T} = \Psi_{i+1}$ which, when applied to Equation (A.17), yields (b).

To establish (c), note that when $P_T \in [\Psi_i, \Psi_{i+1}]$, then $\tilde{P}_{i,T} = P_T$ and thus $R^i_{P\&L} = \overline{R}^i_{P\&L}(P_T)$ with the latter function being given explicitly as follows:

$$\overline{R}^i_{P\&L}(P) = \frac{\left(\sqrt{P} - \sqrt{\Psi_i}\right) + \left(\frac{1}{\sqrt{P}} - \frac{1}{\sqrt{\Psi_{i+1}}}\right) \times P}{\sqrt{\Psi_{i+1}} - \sqrt{\Psi_i}} \tag{A.18}$$

Note that $\overline{R}^i_{P\&L}$ is differentiable with the derivative given explicitly as follows:

$$\frac{d\overline{R}^i_{P\&L}}{dP} = \frac{\frac{1}{\sqrt{P}} - \frac{1}{\sqrt{\Psi_{i+1}}}}{\sqrt{\Psi_{i+1}} - \sqrt{\Psi_i}} \geqslant 0 \tag{A.19}$$

In turn, $P_T \in [\Psi_i, \Psi_{i+1}]$ implies $\frac{1}{\sqrt{1+\Delta}} = \overline{R}^i_{P\&L}(\Psi_i) \leqslant R^i_{P\&L} = \overline{R}^i_{P\&L}(P_T) \leqslant \overline{R}^i_{P\&L}(\Psi_{i+1})$ where $\overline{R}^i_{P\&L}(\Psi_i) \leqslant \overline{R}^i_{P\&L}(P_T) \leqslant \overline{R}^i_{P\&L}(\Psi_{i+1})$ follows from the fact that $\overline{R}$ is a weakly increasing function (i.e., Equation A.19). Finally, using the fact that $\overline{R}^i_{P\&L}(\Psi_i) = \frac{1}{\sqrt{1+\Delta}}$ and $\overline{R}^i_{P\&L}(\Psi_{i+1}) = 1$ we obtain (c). □

A.6 Lemma A.3

Lemma A.3. *Price Interval Contains the Initial Price Level*

For any price interval i such that $P_0 \in [\Psi_i, \Psi_{i+1}]$, the following results hold:

(a) For $P_T \leqslant \Psi_i$, $\frac{P_T}{\Psi_i\sqrt{1+\Delta}} \leqslant R^i_{P\&L} \leqslant \frac{P_T}{\Psi_i}$

(b) For $P_T \geqslant \Psi_{i+1}$, $1 \leqslant R^i_{P\&L} \leqslant \sqrt{1+\Delta}$

(c) For $P_T \in [\Psi_i, \Psi_{i+1}]$, $\frac{1}{\sqrt{1+\Delta}} \leqslant R^i_{P\&L} \leqslant \sqrt{1+\Delta}$

Proof.

As a preliminary step, we define a function, $\Gamma(P, \Psi_i, \Psi_{i+1})$ as follows:

$$\Gamma(P, \Psi_i, \Psi_{i+1}) = \left(\sqrt{P} - \sqrt{\Psi_i}\right) + \left(\frac{1}{\sqrt{P}} - \frac{1}{\sqrt{\Psi_{i+1}}}\right) \times P \tag{A.20}$$

Then, note that:

$$\frac{\partial \Gamma}{\partial P} = \frac{1}{\sqrt{P}} - \frac{1}{\sqrt{\Psi_{i+1}}} \tag{A.21}$$

and thus $P \leqslant \Psi_{i+1}$ implies that $\frac{\partial \Gamma}{\partial P} \geqslant 0$.

We now turn to deriving (a). In particular, whenever $P_T \leqslant \Psi_i$ and $P_0 \in [\Psi_i, \Psi_{i+1}]$, then

$R^i_{P\&L}$ can be written as follows:

$$R^i_{P\&L} = \frac{\left(\frac{1}{\sqrt{\Psi_i}} - \frac{1}{\sqrt{\Psi_{i+1}}}\right) \times P_T}{\Gamma(P_0, \Psi_i, \Psi_{i+1})} \tag{A.22}$$

and thus, when $P_T \leqslant \Psi_i$ the fact that $\frac{\partial \Gamma}{\partial P} \geqslant 0$ for $P \leqslant \Psi_{i+1}$ and $P_0 \in [\Psi_i, \Psi_{i+1}]$ implies that:

$$\frac{\left(\frac{1}{\sqrt{\Psi_i}} - \frac{1}{\sqrt{\Psi_{i+1}}}\right) \times P_T}{\Gamma(\Psi_{i+1}, \Psi_i, \Psi_{i+1})} \leqslant R^i_{P\&L} \leqslant \frac{\left(\frac{1}{\sqrt{\Psi_i}} - \frac{1}{\sqrt{\Psi_{i+1}}}\right) \times P_T}{\Gamma(\Psi_i, \Psi_i, \Psi_{i+1})} \tag{A.23}$$

which, by direct verification, is equivalent to (a):

$$\frac{P_T}{\Psi_i \sqrt{1+\Delta}} \leqslant R^i_{P\&L} \leqslant \frac{P_T}{\Psi_i} \tag{A.24}$$

To prove (b), note that when $P_T \geqslant \Psi_{i+1}$ and $P_0 \in [\Psi_i, \Psi_{i+1}]$, then $R^i_{P\&L}$ can be written as follows:

$$R^i_{P\&L} = \frac{\sqrt{\Psi_{i+1}} - \sqrt{\Psi_i}}{\Gamma(P_0, \Psi_i, \Psi_{i+1})} \tag{A.25}$$

and thus, when $P_T \geqslant \Psi_{i+1}$, the fact that $\frac{\partial \Gamma}{\partial P} \geqslant 0$ for $P_0 \leqslant \Psi_{i+1}$ and $P_0 \in [\Psi_i, \Psi_{i+1}]$ implies:

$$\frac{\sqrt{\Psi_{i+1}} - \sqrt{\Psi_i}}{\Gamma(\Psi_{i+1}, \Psi_i, \Psi_{i+1})} \leqslant R^i_{P\&L} \leqslant \frac{\sqrt{\Psi_{i+1}} - \sqrt{\Psi_i}}{\Gamma(\Psi_i, \Psi_i, \Psi_{i+1})} \tag{A.26}$$

which, by direct verification, is equivalent to (b):

$$1 \leqslant R^i_{P\&L} \leqslant \sqrt{1+\Delta} \tag{A.27}$$

Finally, to establish (c), note that when $P_T \in [\Psi_i, \Psi_{i+1}]$ and $P_0 \in [\Psi_i, \Psi_{i+1}]$, then $R^i_{P\&L}$ can be written as follows:

$$R^i_{P\&L} = \frac{\Gamma(P_T, \Psi_i, \Psi_{i+1})}{\Gamma(P_0, \Psi_i, \Psi_{i+1})} \tag{A.28}$$

and thus, when $P_T \in [\Psi_i, \Psi_{i+1}]$, the fact that $\frac{\partial \Gamma}{\partial P} \geqslant 0$ for $P \leqslant \Psi_{i+1}$ and $P_0 \in [\Psi_i, \Psi_{i+1}]$

implies:

$$\frac{\Gamma(\Psi_i,\Psi_i,\Psi_{i+1})}{\Gamma(\Psi_{i+1},\Psi_i,\Psi_{i+1})} \leqslant R^i_{P\&L} \leqslant \frac{\Gamma(\Psi_{i+1},\Psi_i,\Psi_{i+1})}{\Gamma(\Psi_i,\Psi_i,\Psi_{i+1})} \tag{A.29}$$

which, by direct verification, is equivalent to (c):

$$\frac{1}{\sqrt{1+\Delta}} \leqslant R^i_{P\&L} \leqslant \sqrt{1+\Delta} \tag{A.30}$$

□

A.7 Proof of Proposition 4.3

The first part of the proof, Equation (27), is implied by Lemma A.1. The second part of the proof, Equation (28), is implied by Lemma A.2. □

A.8 Proposition 4.4

By assumption, both price interval i and j satisfy the conditions of Lemma A.1 (i.e., $\Psi_i > \Psi_j \geqslant P_0$). In turn, the first part of this result, Equation (29), follows directly from Lemma A.1. More explicitly, when $P_T < \Psi_j < \Psi_i$, Lemma A.1 (a) implies $R^i_{P\&L} = R^j_{P\&L} = \frac{P_T}{P_0}$; when $P_T \in [\Psi_j, \Psi_{j+1}]$, Lemma A.1 (a) and (c) imply $R^j_{P\&L} \leqslant \frac{P_T}{P_0} = R^i_{P\&L}$; finally, when $P_T > \Psi_{j+1}$, Lemma A.1 (a) - (c) implies $R^j_{P\&L} = \frac{\Psi_j}{P_0}\sqrt{1+\Delta} \leqslant \frac{\Psi_{j+1}}{P_0} \leqslant \min\{\frac{P_T}{P_0}, \frac{\Psi_i}{P_0}\}$. Finally, note that $R^i_{P\&L} \geqslant \min\{\frac{P_T}{P_0}, \frac{\Psi_i}{P_0}\}$ as when $P_T < \Psi_i$ then $R^i_{P\&L} = \frac{P_T}{P_0} = \min\{\frac{P_T}{P_0}, \frac{\Psi_i}{P_0}\}$, when $P_T \in [\Psi_i, \Psi_{i+1}]$ then $R^i_{P\&L} = \frac{\Psi_i}{P_0}\sqrt{1+\Delta} \geqslant \frac{\Psi_i}{P_0} = \min\{\frac{P_T}{P_0}, \frac{\Psi_i}{P_0}\}$, and when $P_T > \Psi_{i+1}$ then $R^i_{P\&L} \geqslant \frac{\Psi_i}{P_0} = \min\{\frac{P_T}{P_0}, \frac{\Psi_i}{P_0}\}$. Thus, we have shown that $R^j_{P\&L} \leqslant \min\{\frac{P_T}{P_0}, \frac{\Psi_i}{P_0}\} \leqslant R^i_{P\&L}$.

For the second part of the result, Equation (27), we derive it explicitly as follows: $\mathbb{P}(\lim_{i\to\infty} R^i_{P\&L} = \frac{P_T}{P_0}) \geqslant \mathbb{P}(\bigcup_{N=1}^{\infty}\bigcap_{i=N}^{\infty}\{R^i_{P\&L} = \frac{P_T}{P_0}\}) \geqslant \mathbb{P}(\bigcup_{N=1}^{\infty}\bigcap_{i=N}^{\infty}\{P_T < \Psi_i\}) = \lim_{N\to\infty}\mathbb{P}(\bigcap_{i=N}^{\infty}\{P_T < \Psi_i\}) = \lim_{N\to\infty}\mathbb{P}(P_T < \Psi_N) = \mathbb{P}(P_T < \infty) = 1$ where the second inequality holds due to Lemma A.1. □

A.9 Proposition 4.5

By assumption, both price intervals i and j satisfy the conditions of Lemma A.2 (i.e., $\Psi_{j+1} < \Psi_{i+1} \leqslant P_0$). In turn, the second part of this result, Equation (31), follows directly from Lemma A.2. More explicitly, when $P_T > \Psi_{i+1}$, Lemma A.2 (b) implies $R^i_{P\&L} = R^j_{P\&L} = 1$; when $P_T \in [\Psi_i, \Psi_{i+1}]$, Lemma A.2 (b) and (c) imply $R^i_{P\&L} \leqslant 1 = R^j_{P\&L}$; when $P_T \in [\Psi_{j+1}, \Psi_i]$, Lemma A.2 (a) and (c) imply $R^i_{P\&L} = \frac{P_T}{\Psi_i\sqrt{1+\Delta}} \leqslant \frac{1}{\sqrt{1+\Delta}} \leqslant 1 = R^j_{P\&L}$; finally, when $P_T < \Psi_{j+1}$, Lemma A.2 (a) implies $R^i_{P\&L} = \frac{P_T}{\Psi_i\sqrt{1+\Delta}} \leqslant \frac{P_T}{\Psi_j\sqrt{1+\Delta}} = R^j_{P\&L}$.

For the second part of the result, Equation (28), we derive it explicitly as follows: $\mathbb{P}(\lim_{i\to-\infty} R^i_{P\&L} = 1) \geqslant \mathbb{P}(\bigcup_{N=1}^{\infty}\bigcap_{i=N}^{\infty}\{R^{-i}_{P\&L} = 1\}) \geqslant \mathbb{P}(\bigcup_{N=1}^{\infty}\bigcap_{i=N}^{\infty}\{P_T > \Psi_{-i+1}\}) = \lim_{N\to\infty}\mathbb{P}(\bigcap_{i=N}^{\infty}\{P_T > \Psi_{-i+1}\}) = \lim_{N\to\infty}\mathbb{P}(\{P_T > \Psi_{-N+1}\}) = \mathbb{P}(P_T > -\infty) = 1$ where the second inequality holds due to Lemma A.2. □

A.10 Lemma A.4

Lemma A.4. *Limiting Fee Level*

For any price level $P > 0$, let $p := \log(P)$. Then, the following result holds:

$$\lim_{\Delta\to 0^+}\frac{1}{\Delta}\int_0^T \mathbb{Q}(P_t \in [\Psi_{i(\Delta,P)}, \Psi_{i(\Delta,P)+1}])\ dt = \int_0^T f(p,t)\ dt \tag{A.31}$$

where $i(\Delta, P)$ is such that $P \in [\Psi_{i(\Delta,P)}, \Psi_{i(\Delta,P)+1}]$ for all $\Delta > 0$.

Proof.

Let $\psi_i(\Delta, P) := \log(\Psi_{i(\Delta,P)})$ and let $\delta := \log(1 + \Delta)$. Then:

$$\int_0^T \mathbb{Q}(P_t \in [\Psi_{i(\Delta,P)}, \Psi_{i(\Delta,P)+1}])\ dt = \int_0^T \int_{\psi_i(\Delta,P)}^{\psi_i(\Delta,P)+\delta} f(p,t)\ dp\ dt$$

In turn, continuity of $f(p,t)$ in its first argument implies:

$$\frac{1}{\delta}\int_0^T \mathbb{Q}(P_t \in [\Psi_{i(\Delta,P)}, \Psi_{i(\Delta,P)+1}])\ dt = \int_0^T f(p_{i(\Delta,P)}, t)\ dt$$

where $p_{i(\Delta,P)} \in [\psi_i(\Delta, P), \psi_i(\Delta, P) + \delta] \subseteq [p - \delta, p + \delta]$.

Finally, note that $\Delta \to 0^+ \Leftrightarrow \delta \to 0^+$ and thus:

$$\begin{aligned}
&\lim_{\Delta\to 0^+} \frac{1}{\Delta}\int_0^T \mathbb{Q}(P_t \in [\Psi_{i(\Delta,P)}, \Psi_{i(\Delta,P)+1}])\ dt \\
&= \lim_{\Delta\to 0^+} \frac{\Delta}{\log(1+\Delta)} \times \lim_{\delta\to 0^+} \frac{1}{\delta}\int_0^T \mathbb{Q}(P_t \in [\Psi_{i(\Delta,P)}, \Psi_{i(\Delta,P)+1}])\ dt \\
&= \lim_{\delta\to 0^+} \int_0^T f(p_{i(\Delta,P)}, t)\ dt \\
&= \int_0^T \lim_{\delta\to 0^+} f(p_{i(\Delta,P)}, t)\ dt \\
&= \int_0^T f(p, t)\ dt
\end{aligned}$$

The second-to-last line follows from the Bounded Convergence Theorem, whereas the last line follows from continuity of $f(p,t)$ and $p_{i(\Delta,P)} \in [p-\delta, p+\delta]$ for all δ. To provide more detail on the former, note that $\limsup_{\Delta\to 0^+} \int_0^T |f(p_{i(\Delta,P)}, t)|\ dt < S \times T < \infty$ where $S := \max\{f(\rho,\tau) : \rho \in [p-\varepsilon, p+\varepsilon], \tau \in [0,T]\} < \infty$ for any $\varepsilon > 0$ and where the existence of a finite maximum follows from continuity of $f(p,t)$. □

A.11 Lemma A.5

Lemma A.5. *Limiting Ex-Fee Portfolio Value*

For any price level $P > 0$, the following result holds for any $t \in [0,1]$:

$$\lim_{\Delta\to 0^+} \frac{\Pi^\star_{i(\Delta,P),t}}{L^\star_i\left(\sqrt{1+\Delta}-1\right)} = \begin{cases} \frac{P_t}{\sqrt{P}} & \text{if } P_t < P \\ \sqrt{P} & \text{if } P_t \geqslant P \end{cases} \tag{A.32}$$

where $i(\Delta, P)$ is such that $P \in [\Psi_{i(\Delta,P)}, \Psi_{i(\Delta,P)+1}]$ for all $\Delta > 0$.

Proof.

For $P_t < P$, the result arises by direct verification, applying Equation (20) to Equation (8):

$$\lim_{\Delta\to 0^+} \frac{\Pi^\star_{i(\Delta,P),t}}{L_i^\star\left(\sqrt{1+\Delta}-1\right)} = \lim_{\Delta\to 0^+} \frac{\left(\frac{1}{\sqrt{\Psi_{i(\Delta,P)}}}-\frac{1}{\sqrt{\Psi_{i(\Delta,P)+1}}}\right)\times P_t}{\sqrt{1+\Delta}-1} = \lim_{\Delta\to 0^+} \frac{P_t}{\sqrt{\Psi_{i(\Delta,P)+1}}} = \frac{P_t}{\sqrt{P}}$$

For $P_t > P$, the result also arises directly, by applying Equation (20) to Equation (8):

$$\lim_{\Delta\to 0^+} \frac{\Pi^\star_{i(\Delta,P),t}}{L_i^\star\left(\sqrt{1+\Delta}-1\right)} = \lim_{\Delta\to 0^+} \frac{\sqrt{\Psi_{i(\Delta,P)+1}}-\sqrt{\Psi_{i(\Delta,P)}}}{\sqrt{1+\Delta}-1} = \lim_{\Delta\to 0^+} \sqrt{\Psi_{i(\Delta,P)}} = \sqrt{P}$$

For the case of $P_t = P$, it is useful to define $\tilde{\Pi}(X, \Psi_i, \Psi_{i+1})$ as follows:

$$\tilde{\Pi}(X, \Psi_i, \Psi_{i+1}) := \left(\sqrt{X} - \sqrt{\Psi_i}\right) + \left(\frac{1}{\sqrt{X}} - \frac{1}{\sqrt{\Psi_{i+1}}}\right) \times X \tag{A.33}$$

Direct inspection reveals $\frac{\Pi^\star_{i(\Delta,P),t}}{L_i^\star} = \tilde{\Pi}(P_t, \Psi_{i(\Delta,P)}, \Psi_{i(\Delta,P)+1})$ and also that $\frac{\partial\tilde{\Pi}}{\partial X} \geqslant 0$ whenever $X \leqslant \Psi_{i+1}$. In turn, we have the following result:

$$\tilde{\Pi}(\Psi_{i(\Delta,P)}, \Psi_{i(\Delta,P)}, \Psi_{i(\Delta,P)+1}) \leqslant \frac{\Pi^\star_{i(\Delta,P),t}}{L_i^\star} \leqslant \tilde{\Pi}(\Psi_{i(\Delta,P)+1}, \Psi_{i(\Delta,P)}, \Psi_{i(\Delta,P)+1}) \tag{A.34}$$

Moreover, applying $\tilde{\Pi}(\Psi_{i(\Delta,P)}, \Psi_{i(\Delta,P)}, \Psi_{i(\Delta,P)+1}) = \left(\frac{1}{\sqrt{\Psi_{i(\Delta,P)}}} - \frac{1}{\sqrt{\Psi_{i(\Delta,P)+1}}}\right) \times \Psi_{i(\Delta,P)} = \left(\sqrt{1+\Delta}-1\right)\frac{\Psi_{i(\Delta,P)}}{\sqrt{\Psi_{i(\Delta,P)+1}}}$ and $\tilde{\Pi}(\Psi_{i(\Delta,P)+1}, \Psi_{i(\Delta,P)}, \Psi_{i(\Delta,P)+1}) = \sqrt{\Psi_{i(\Delta,P)+1}} - \sqrt{\Psi_{i(\Delta,P)}} = \left(\sqrt{1+\Delta}-1\right)\sqrt{\Psi_{i(\Delta,P)}}$ to Equation (A.34) and taking the limit as $\Delta \to 0^+$ completes the proof as follows:

$$\sqrt{P} = \lim_{\Delta\to 0^+} \frac{\Psi_{i(\Delta,P)}}{\sqrt{\Psi_{i(\Delta,P)+1}}} \leqslant \lim_{\Delta\to 0^+} \frac{\Pi^\star_{i(\Delta,P),t}}{L_i^\star\left(\sqrt{1+\Delta}-1\right)} \leqslant \lim_{\Delta\to 0^+} \sqrt{\Psi_{i(\Delta,P)}} = \sqrt{P} \tag{A.35}$$

□

A.12 Lemma A.6

Lemma A.6. *Limiting Ex-Fee Portfolio Return*

For any price level $P > 0$, the following result holds:

$$\lim_{\Delta \to 0^+} R_{P\&L}^{i(\Delta,P)} = \frac{\min\{P_T, P\}}{\min\{P_0, P\}} = \frac{P_T - (P_T - P)^+}{\min\{P_0, P\}} = \frac{P - (P - P_T)^+}{\min\{P_0, P\}} \tag{A.36}$$

where $i(\Delta, P)$ is such that $P \in [\Psi_{i(\Delta,P)}, \Psi_{i(\Delta,P)+1}]$ for all $\Delta > 0$.

Proof.

Equation (7) yields:

$$R_{P\&L}^{i(\Delta,P)} = \frac{\Pi^\star_{i(\Delta,P),T}}{\Pi^\star_{i(\Delta,P),0}} = \frac{\frac{\Pi^\star_{i(\Delta,P),T}}{L_i^\star\left(\sqrt{1+\Delta}-1\right)}}{\frac{\Pi^\star_{i(\Delta,P),0}}{L_i^\star\left(\sqrt{1+\Delta}-1\right)}}$$

Taking $\Delta \to 0^+$ and applying Lemma A.5 then implies the result:

$$\lim_{\Delta \to 0^+} R_{P\&L}^{i(\Delta,P)} = \lim_{\Delta \to 0^+} \frac{\frac{\Pi^\star_{i(\Delta,P),T}}{L_i^\star\left(\sqrt{1+\Delta}-1\right)}}{\frac{\Pi^\star_{i(\Delta,P),0}}{L_i^\star\left(\sqrt{1+\Delta}-1\right)}} = \frac{\lim_{\Delta \to 0^+} \frac{\Pi^\star_{i(\Delta,P),T}}{L_i^\star\left(\sqrt{1+\Delta}-1\right)}}{\lim_{\Delta \to 0^+} \frac{\Pi^\star_{i(\Delta,P),0}}{L_i^\star\left(\sqrt{1+\Delta}-1\right)}} = \frac{\min\{P_T,P\}}{\min\{P_0,P\}}$$

where the last equality follows by direct verification. □

A.13 Lemma A.7

Lemma A.7. *Limiting Ex-Fee Expected Return*

For any price level $P > 0$, the following result holds:

$$\lim_{\Delta \to 0^+} \mathbb{E}^{\mathbb{Q}}[R_{P\&L}^{i(\Delta,P)}] = \frac{P_0 e^{rT} - e^{rT}\mathcal{C}(P,T)}{\min\{P_0, P\}} \tag{A.37}$$

where $i(\Delta, P)$ is such that $P \in [\Psi_{i(\Delta,P)}, \Psi_{i(\Delta,P)+1}]$ for all $\Delta > 0$ and $\mathcal{C}(K, \tau) := e^{-rT}\mathbb{E}^{\mathbb{Q}}[(P_\tau - K)^+]$ refers to the price of a European call option with ETH-USDC as the underlying, K as the strike price and τ as the time to maturity.

Proof.

Lemmas A.1 - A.3 imply that for all $\varepsilon > 0$:

$$\limsup_{\Delta \to 0^+} R^i_{P\&L} \leqslant (1+\varepsilon) \times \max\{\frac{P}{P_0}, \sqrt{1+\Delta}\} < \infty \tag{A.38}$$

and thus the tail of $\{R^i_{P\&L}\}_\Delta$ is bounded so that the Bounded Convergence Theorem implies:

$$\lim_{\Delta \to 0^+} \mathbb{E}^{\mathbb{Q}}[R^{i(\Delta,P)}_{P\&L}] = \mathbb{E}^{\mathbb{Q}}[\lim_{\Delta \to 0^+} R^{i(\Delta,P)}_{P\&L}] \tag{A.39}$$

Moreover, Lemma A.6 further implies:

$$\lim_{\Delta \to 0^+} R^{i(\Delta,P)}_{P\&L} = \frac{\min\{P_T, P\}}{\min\{P_0, P\}} = \frac{P_T - (P_T - P)^+}{\min\{P_0, P\}} \tag{A.40}$$

where the second equality follows from $\min\{x, y\} = x - (x-y)^+$.

Finally, Equation (5) and $\mathbb{E}[e^{\frac{1}{2}\int_0^T \sigma_t^2 dW_t}] < \infty$ imply that $M_t := e^{-rt}P_t$ is a $\mathbb{Q}$-martingale and thus $\mathbb{E}^{\mathbb{Q}}[P_T] = e^{rT}\mathbb{E}^{\mathbb{Q}}[M_T] = e^{rT}M_0 = e^{rT}P_0$. In turn, Equations (A.39) and (A.40) imply the desired result:

$$\lim_{\Delta \to 0^+} \mathbb{E}^{\mathbb{Q}}[R^{i(\Delta,P)}_{P\&L}] = \frac{\mathbb{E}^{\mathbb{Q}}[P_T] - \mathbb{E}^{\mathbb{Q}}[(P_T - P_0)^+]}{\min\{P_0, P\}} = \frac{P_0 e^{rT} - e^{rT}\mathcal{C}(P,T)}{\min\{P_0, P\}} \tag{A.41}$$

$\square$

A.14 Proof of Proposition 4.7

This result follows directly from Lemmas A.4 and A.7.

A.15 Proof of Proposition 4.8

Proof.

This follows directly from Lemma A.6. More explicitly:

$$\lim_{\Delta \to 0^+} R_{P\&L}^{i(\Delta,P)} = \frac{\min\{P_T, P\}}{\min\{P_0, P\}} = \frac{P_T - (P_T - P)^+}{P_0 - (P_0 - P)^+}$$

where the first equality is established by Lemma A.6 and the second equality follows from the identity $\min\{x, y\} = x - (x - y)^+$. □